1. Given 2 regular languages L_1 and L_2, we want to prove $L_1 \cap L_2$ is also a regular language.

By definition of regularity, there exists DFAs M_1 and M_2 such that M_1 recognizes L_1 and M_2 recognizes L_2.

Let $M_1 = (Q_1, \Sigma_1, \delta_1, q_1, F_1)$

And let $M_2 = (Q_2, \Sigma_2, \delta_2, q_2, F_2)$

Let's construct a new machine

$M = (Q, \Sigma, \delta, q_0, F)$ such that

$Q = Q_1 \times Q_2$

$\Sigma = \Sigma_1 \cap \Sigma_2$

$\delta((q,q'), a) = (\delta_1(q,a), \delta_2(q',a))$

$q_0 = (q_1, q_2)$

$F = \{(q,q') \mid q \in F_1 \land q' \in F_2\}$

Let's now confirm that our machine recognizes $L_1 \cap L_2$.

We want $(L_1 \cap L_2)(M) = (L_1 \cap L_2)$

Notice $\delta^*((q,q'), w) = (\delta_1^*(q,w), \delta_2^*(q',w))$

$w \in L_1$ iff $\delta_1^*(q_1, w) \in F_1$ and

$w \in L_2$ iff $\delta_2^*(q_2, w) \in F_2$

So if $w \in L_1 \cap L_2$, then $\delta_1^*(q_1, w) \in F_1$ and $\delta_2^*(q_2, w) \in F_2$

And thus $(\delta_1^*(q_1,w), \delta_2^*(q',w)) \in F$

But $(\delta_1^*(q_1,w), \delta_2^*(q',w)) = \delta^*((q_1,q_2), w)$

So M accepts all $w \in L_1 \cap L_2$

Now if $w \notin L_1 \cap L_2$ then $w \notin L_1 \lor w \notin L_2$,

so either $\delta_1^*(q_1, w) \notin F_1$ or $\delta_2^*(q_2, w) \notin F_2$

thus $(\delta_1^*(q_1, w), \delta_2^*(q_2, w)) \notin F$

And since $(\delta_1^*(q_1,w), \delta_2^*(q',w)) = \delta^*((q_1,q_2), w)$,

$w \notin L_1 \cap L_2 \Rightarrow M$ does not accept w

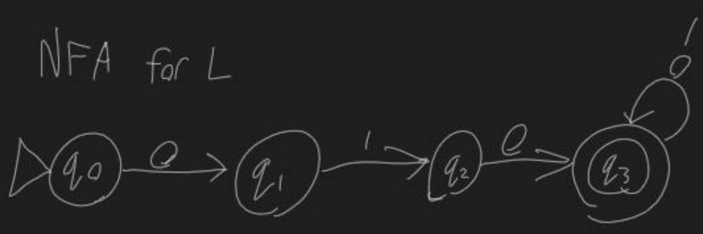
M recognizes $L_1 \cap L_2$ and it is a DFA therefore $L_1 \cap L_2$ is regular.

QED

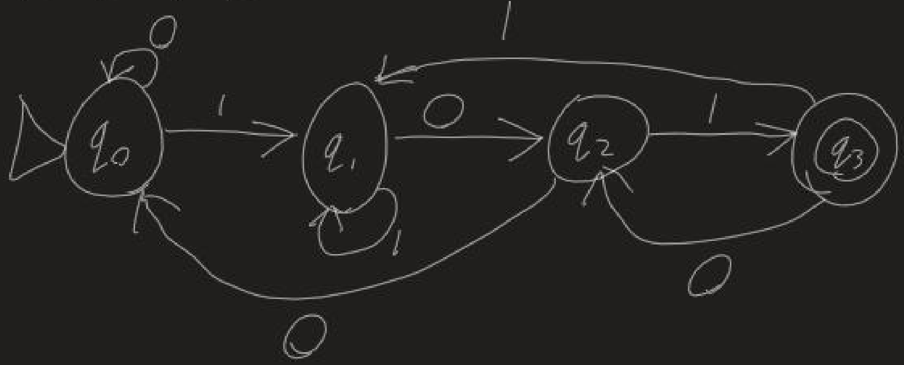
2. $L = \{w \mid w \text{ begins with } 010\}$

$L' = \{w \mid w \text{ ends with } 101\}$

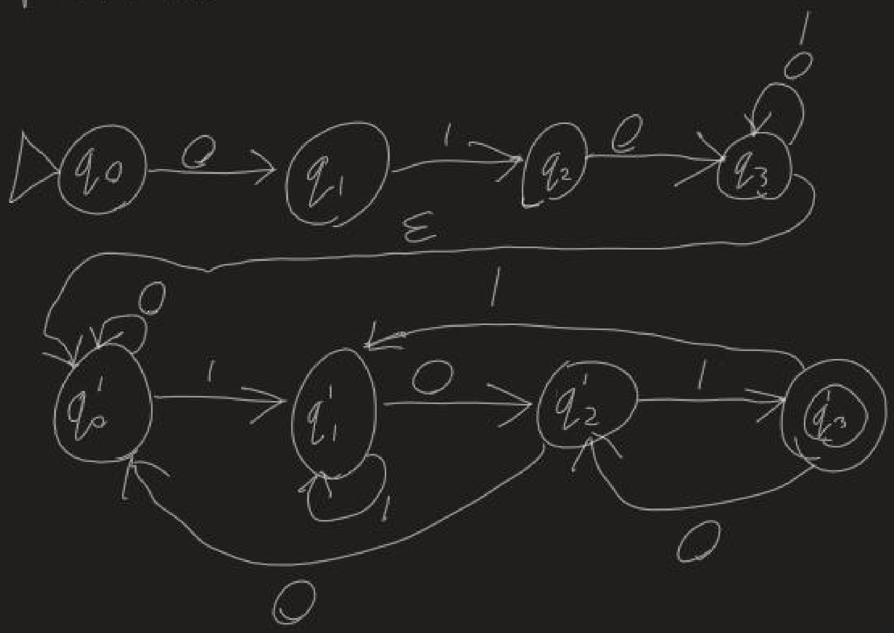
NFA for L



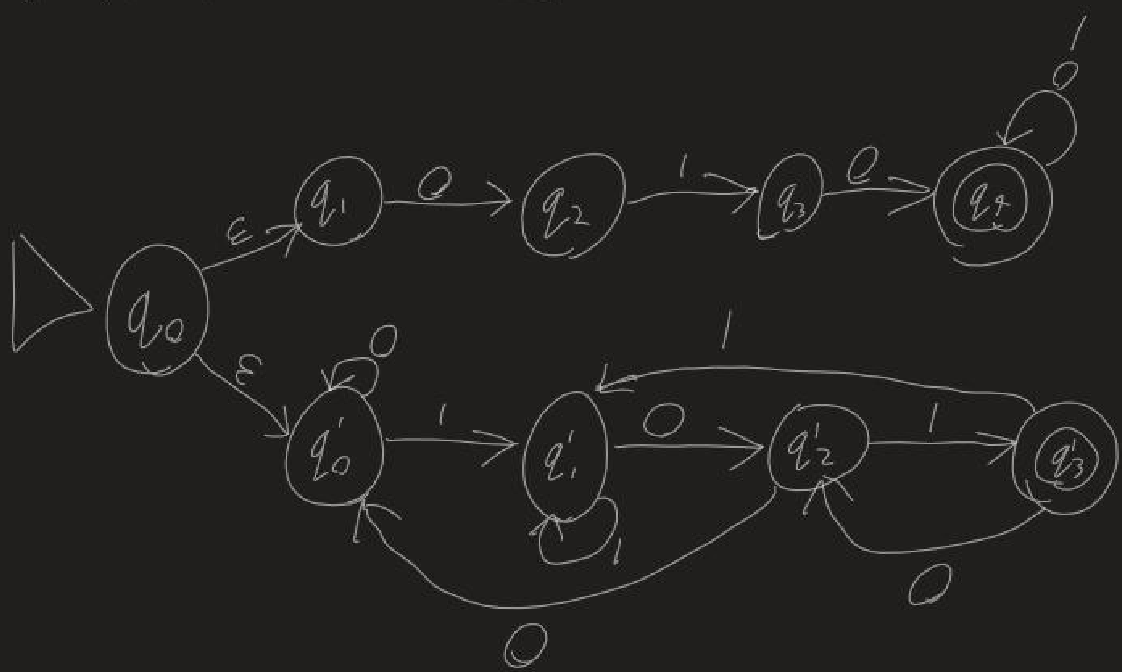
NFA for L'



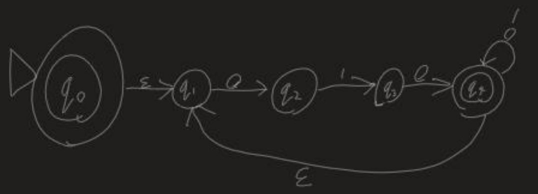
NFA for LL'



NFA for LUL'



NFA for L^*



3.

CurrentStates	NextStates	
$\{q_0\}$	$\emptyset$	
	$\{q_1, q_0'\}$	$\delta(q_0, \varepsilon) = \{q_1, q_0'\}$
$\{q_1, q_0'\}$	$\emptyset$	$\cdot\ \delta(q_1, \varepsilon) = \delta(q_0', \varepsilon) = \delta(q_1, 1) = \emptyset$
	$\{q_1'\}$	$\delta(q_0', 1) = \{q_1'\}$
$\{q_1'\}$	$\emptyset$	$\cdot\ \delta(q_1', \varepsilon) = \emptyset$
	$\{q_2'\}$	$\delta(q_1', 0) = \{q_2'\}$
$\{q_2'\}$	$\emptyset$	$\cdot\ \delta(q_2', \varepsilon) = \emptyset$
	$\{q_3'\}$	$\delta(q_2', 1) = \{q_3'\}$
$\{q_3'\}$	$\emptyset$	$\cdot\ \delta(q_3', \varepsilon) = \emptyset$
	$\{q_1'\}$	$\delta(q_3', 1) = \{q_1'\}$
		We can now stop checking ε transitions for q_0', q_1', q_2', and q_3' since they all yield $\emptyset$
$\{q_1'\}$	$\{q_2'\}$	$\delta(q_1', 0) = \{q_2'\}$
$\{q_2'\}$	$\{q_3'\}$	$\delta(q_2', 1) = \{q_3'\}$
$\{q_3'\}$	$\{q_2'\}$	$\delta(q_3', 0) = \{q_2'\}$
$\{q_2'\}$	$\emptyset$	

CurrentStates does not contain an accept state after processing $w = 1011010$, thus w is not a member of the language.

4. Let our NFA for $L \cup L'$ be $N=(Q, \Sigma, \delta, q_0, F)$

We will construct a DFA, D, $D=(Q', \Sigma, \delta', q_0', F')$

Let $Q' = \mathcal{P}(Q)$

$\forall R \in Q' (\forall a (\delta'(R,a) = \{q \in Q \mid q \in E(\delta(r,a)) \text{ for an } r \in R\}))$

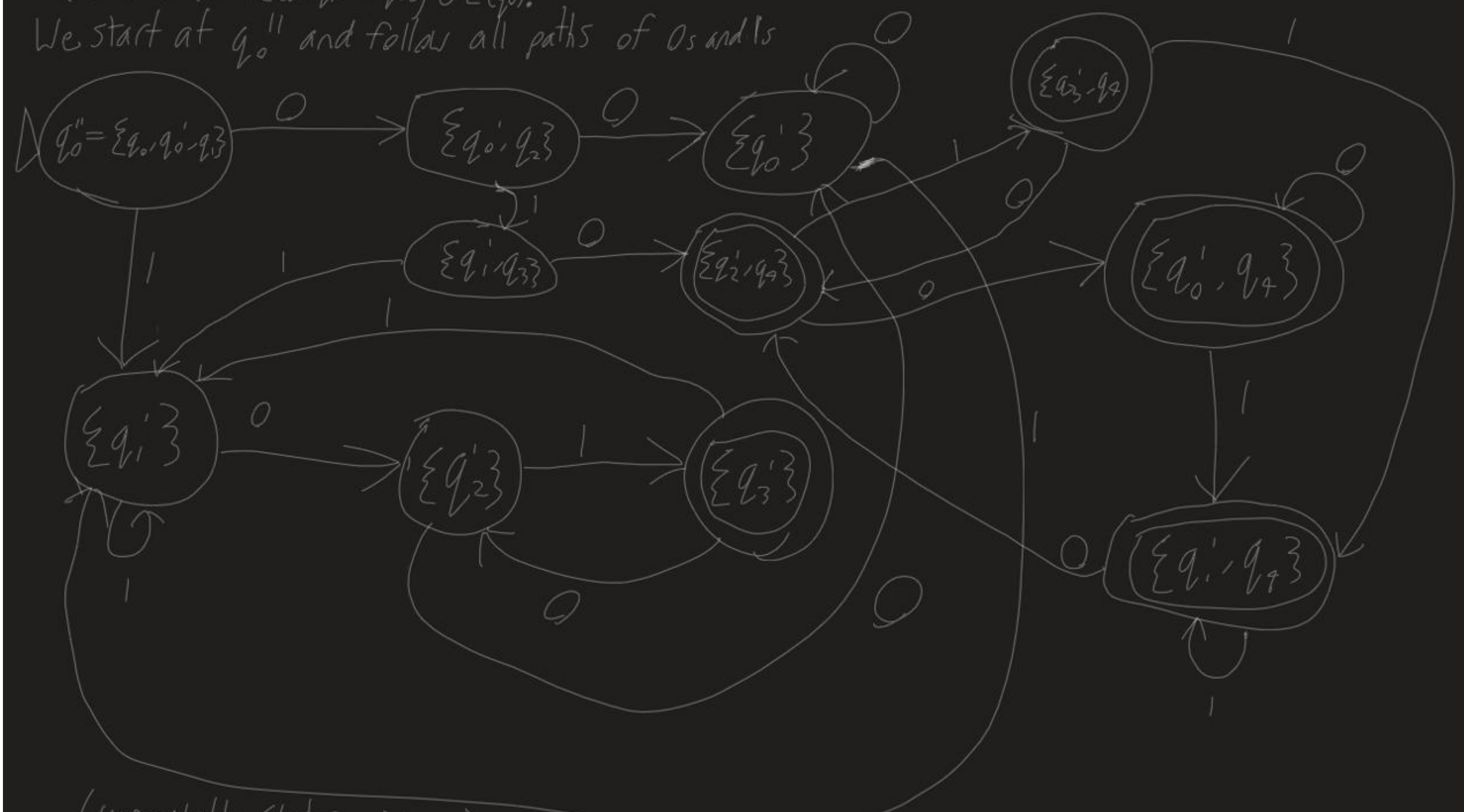
Let $q_0'' = E(q_0)$

Let $F' = \{R \in Q' \mid R \text{ contains an accept state of } N\}$

Many states in the DFA are unreachable, however.

Since N only has one node with multiple outgoing edges, and for every other state, q, $E(q)=\emptyset$, and removing that node would result in a graph that is not connected, we only have to consider the states $(\{q_1, q_2, q_3, q_4\} \times \{q_0', q_1', q_2', q_3'\}) \cup E(q_0)$.

We start at q_0'' and follow all paths of 0s and 1s



(unreachable states not drawn)

Now we want to minimize the DFA.

$(q_0'', \{q_3'\})$ = distinguishable

$(q_0'', \{q_2', q_4\})$ = distinguishable

$(q_0'', \{q_3', q_4\})$ = distinguishable

$(q_0'', \{q_0', q_4\})$ = distinguishable

$(q_0'', \{q_1', q_4\})$ = distinguishable

$(\{q_1'\}, \{q_3'\})$ = distinguishable

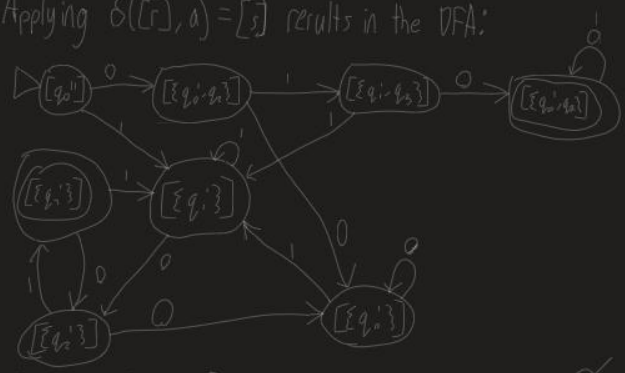
$\vdots$

$(\delta(q_0'', 0) = \{q_0', q_2\}, \delta(\{q_0', q_4\}, 0) = \{q_0', q_4\})$ = distinguishable because $(\{q_0', q_2\}, \{q_0', q_4\})$ is distinguishable

$\vdots$

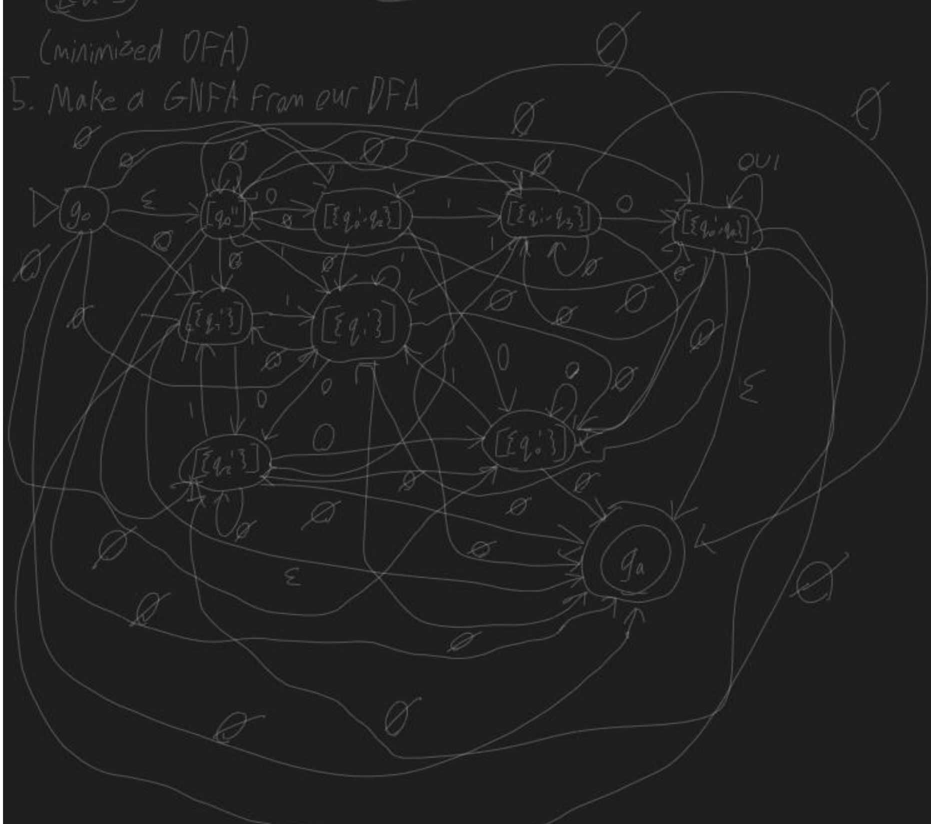
End up with $[q_0'']$, $[\{q_1'\}]$, $[\{q_0', q_2\}]$, $[\{q_2'\}]$, $[\{q_1', q_3\}]$,
$[\{q_0'\}]$, $[\{q_3'\}]$, $[\{q_0', q_4\}]$

Applying $\delta([r], a) = [s]$ results in the DFA:



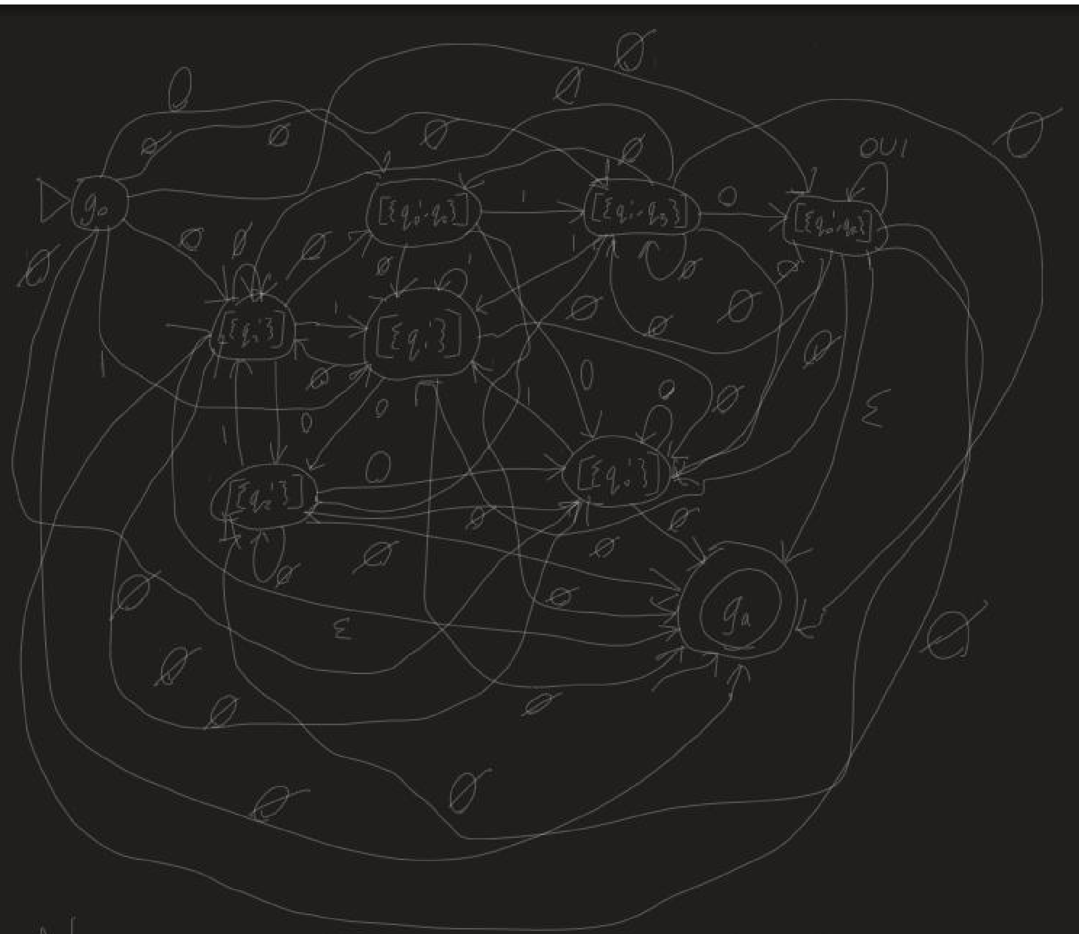
(minimized DFA)

5. Make a GNFA from our DFA

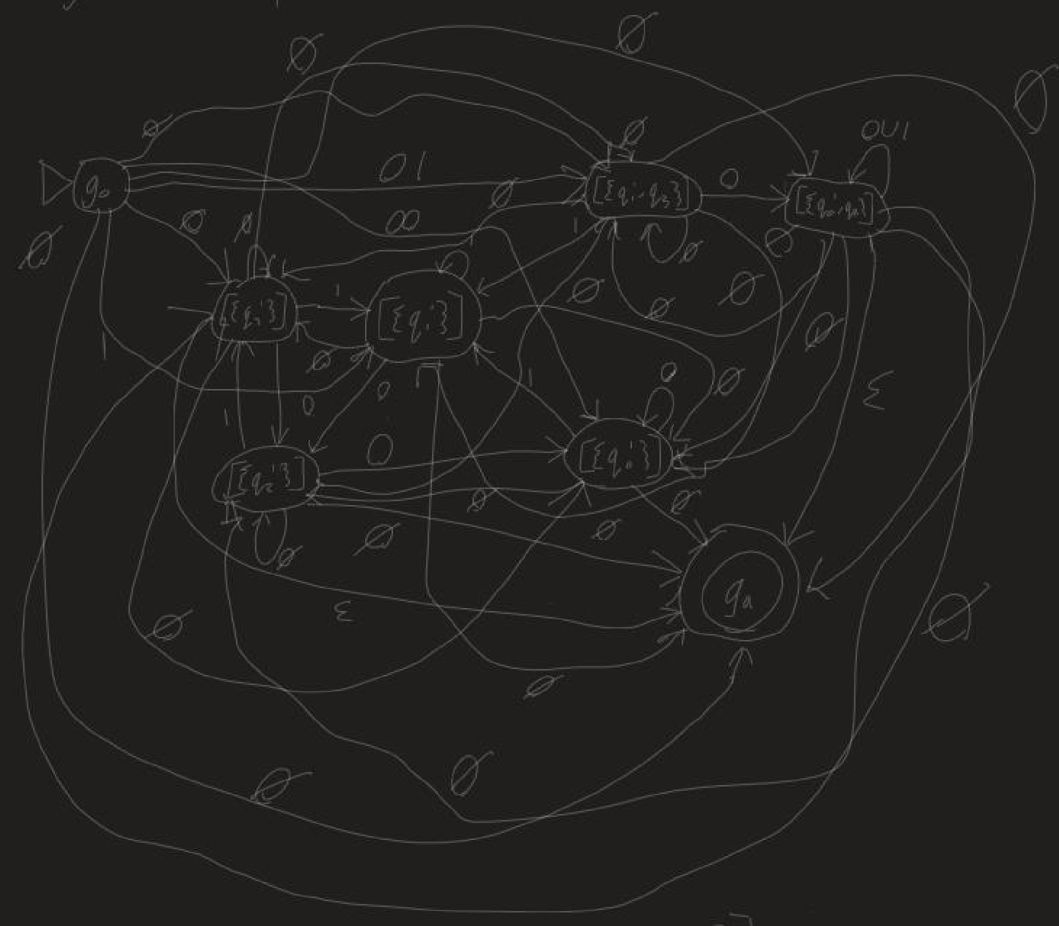


Removing $[q_0'']$ by computing length 2 path through $[q_0'']$
between every start node u and end node v
and unioning it with the length 1 path from u to v

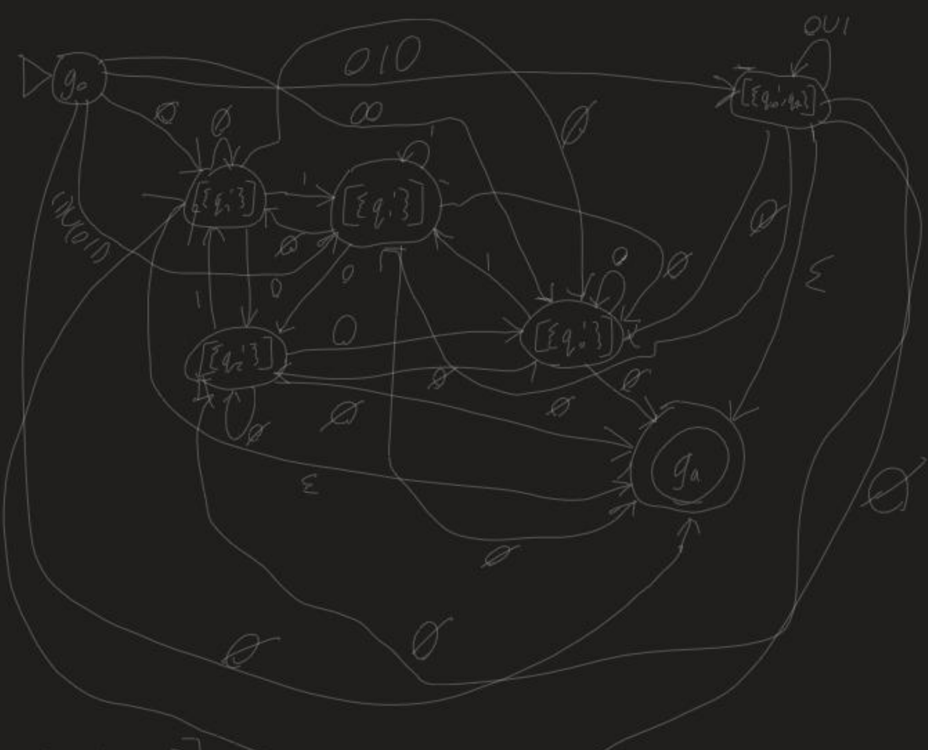
I will evaluate $X \cup \emptyset$ as X and $X\emptyset$ as $\emptyset$ and εa as a



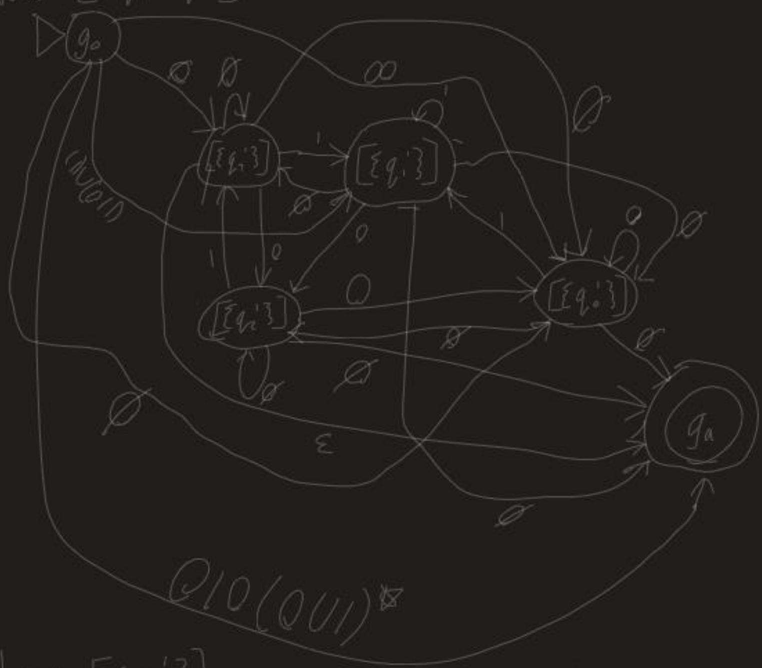
Now we will remove the $[\{q_0, q_2\}]$ node using the same process



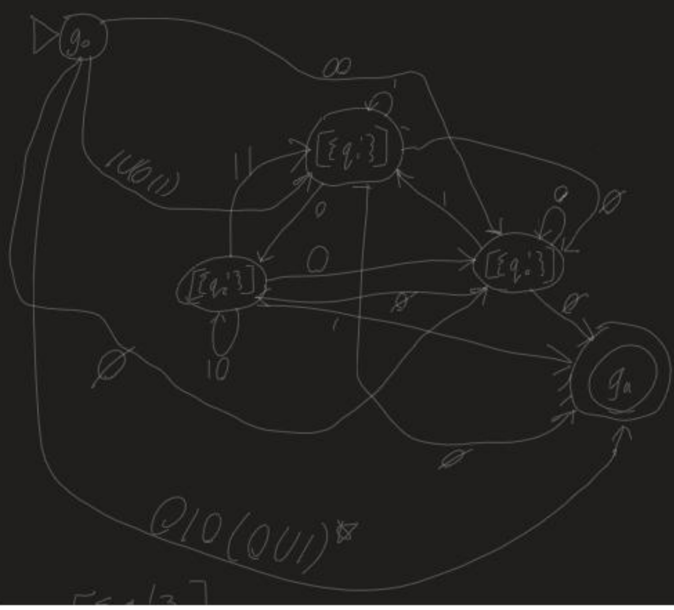
Now we will remove $[\{q_1, q_3\}]$ using the same process



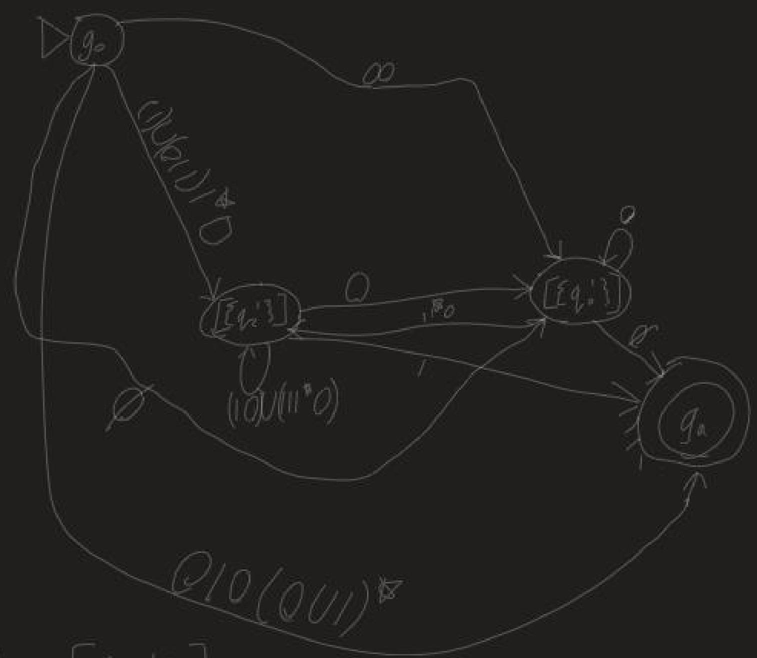
Now [{q0', q4}]



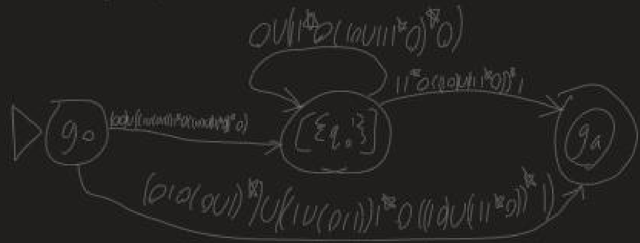
Now [{q3'}]



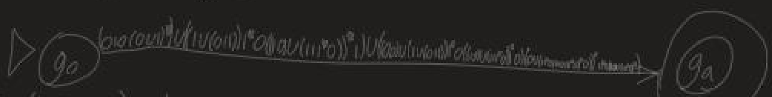
Now $[\{q_1'\}]$



Now $[\{q_2'\}]$



And finally $[\{q_0'\}]$



6. Let $G=(V,\Sigma,P,S)$ where

$V=\{[q_0''], [\{q_0', q_2\}], [\{q_1', q_2\}], [\{q_0', q_2\}], [\{q_1'\}], [\{q_2'\}], [\{q_1'\}], [\{q_0'\}]\}$

$P=\{[q_0''] \rightarrow 0[\{q_0', q_1\}] \mid 1[\{q_1\}],$

$[\{q_0', q_1\}] \rightarrow 0[\{q_0'\}] \mid 1[\{q_1', q_2\}],$

$[\{q_1', q_1\}] \rightarrow 0[\{q_0', q_2\}] \mid 1[\{q_1'\}],$

$[\{q_2', q_1\}] \rightarrow 0[\{q_0', q_2\}] \mid 1[\{q_2', q_1\}] \mid \varepsilon,$

$[\{q_1'\}] \rightarrow 0[\{q_2'\}] \mid 1[\{q_1'\}] \mid \varepsilon,$

$[\{q_1'\}] \rightarrow 0[\{q_1'\}] \mid 1[\{q_2'\}],$

$[\{q_2'\}] \rightarrow 0[\{q_0'\}] \mid 1[\{q_1'\}],$

$[\{q_0'\}] \rightarrow 0[\{q_0'\}] \mid 1[\{q_1'\}]$

$\}$

$S=[q_0'']$

7. Let $h(0)=zero,\ h(1)=\varepsilon,\ h(2)=\varepsilon$

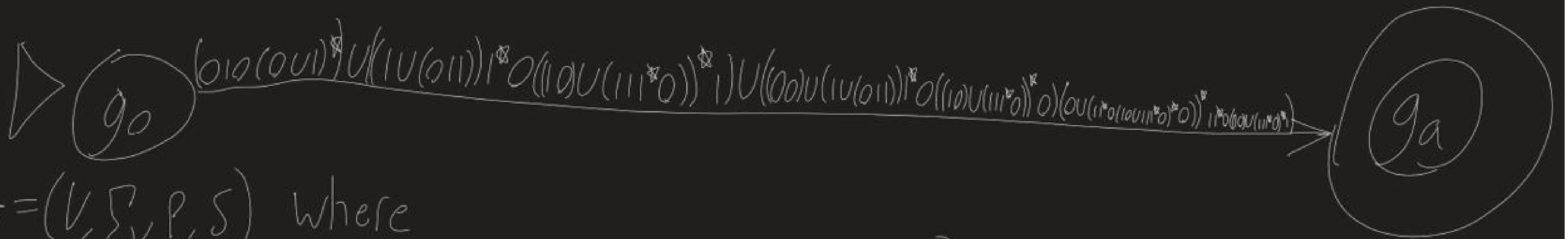
Regex for L: $((1\cup 2)^*0(1\cup 2)^*0(1\cup 2)^*0)^*(1\cup 2)^*$

Construct L' by applying h on regex for L

$((h(1)\cup h(2))^*h(0)(h(1)\cup h(2))^*h(0)(h(1)\cup h(2))^*h(0))^*(h(1)\cup h(2))^*$

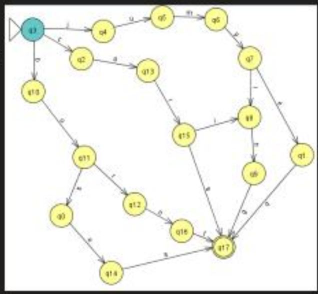
$(\varepsilon^*zero\,\varepsilon^*zero\,\varepsilon^*zero)^*\varepsilon^*$

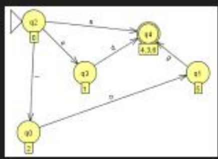
$(zero\,zero\,zero)^*$



$G = (V, \Sigma, P, S)$ where

8.

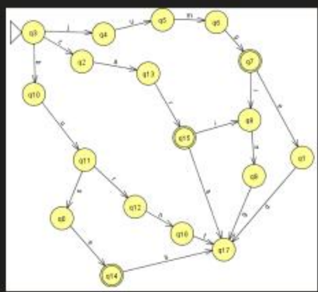
A= 

B= 

We want i s.t. $\delta^*(q_i, w) \in F$ and $w \in L(M')$

All q_i satisfying above condition should be only accept states in $\frac{A}{B}$.

That's q_7, q_{14}, and q_{15}

Thus $\frac{A}{B}$ = 

9.

Proof by induction

Basis step: Level 0 heading

$L_{\leq 0} = \{ =^0 w =^0 \mid w \in \{a, b, c, \ldots, z\}^* \}$

$= \{ w \mid w \in \{a, b, c, \ldots, z\}^* \}$

This language is regular because

$L((a \cup b \cup \ldots \cup z)^*) = L_{\leq 0}$

Inductive Step:

Assume $L_{\leq n}$ is regular

Let R be the regular expression that describes $L_{\leq n}$.

Then The regular expression for L_{n+1} is $=^{n+1}(a \cup b \cup c \cup \ldots \cup z)^* =^{n+1}$

So the regular expression describing $L_{\leq n+1}$ is $(=^{n+1}(a \cup b \cup \ldots \cup z)^* =^{n+1}) \cup R$

So $L_{\leq n+1}$ is regular

We have proven $L_{\leq 0}$ is regular and $L_{\leq n}$ being regular implies $L_{\leq n+1}$ is regular.

Therefore, $L_{\leq n}$ is regular for all $n \in \mathbb{N}$.

QED

10.

Assume $\bigcup_{n\in\mathbb{N}} L_{\le n}$ is regular,

then it has a pumping length p.

Consider $w = =^p a =^p$, $w \in \bigcup_{n\in\mathbb{N}} L_{\le n}$

Using the pumping lemma splitting $w=xyz$

$|xy| \le p$ so $x = =^i$ and $y = =^j$ s.t. $j>0$,

and $z = =^{p-(i+j)} a =^p$,

xy^2z must be in $\bigcup_{n\in\mathbb{N}} L_{\le n}$

by the pumping lemma,

but $xy^2z = =^i =^{2j} =^{p-(i+j)} a =^p$,

which is equivalent to $=^{p+i+2j-(i+j)} a =^p$,

and $p+i+2j-(i+j) = p+j > p$ because $j>0$

so xy^2z is not in the language.

Thus we have a contradiction and $\bigcup_{n\in\mathbb{N}} L_{\le n}$

is not regular.

QED