Review

For the linear equation with constant coefficients,

\[ ay'' + by' + cy = 0, \]

when the roots of the characteristic equation are equal, the general solution is

\[ y = Ae^{\lambda t} + Bte^{\lambda t}. \]

For the linear homogeneous equation with not-necessarily-constant coefficients, if we have one solution \( u \), we can find the general solution \( Au + Bvu \) using the method of reduction of order.

We look for a solution \( y = vu \), and when we plug this into the equation, the coefficients of \( v \) will cancel out and leave a first-order equation for \( v' \).

Solve that equation, and then integrate to find \( v \).
The non-homogeneous equation

Consider the non-homogeneous second-order equation with constant coefficients:

\[ ay'' + by' + cy = F(t). \]

▶ The difference of any two solutions is a solution of the homogeneous equation.

▶ Suppose we have one solution \( u \). Then the general solution is \( u \) plus the general solution of the homogeneous equation.

▶ Proof, let \( y \) be any solution. Then \( y - u \) solves the homogeneous equation, so \( y = u + \) a solution of the homogeneous equation.
The non-homogeneous equation

- Suppose we have one solution $u$. Then the general solution is $u$ plus the general solution of the homogeneous equation.

- So, solving the equation boils down to finding just one solution.

- But there is no foolproof method for doing that (for any arbitrary right-hand side $F(t)$).

- We can do it in some useful common cases.
Constant coefficients or not?

- What we did up to now works for any second-order linear equation.
- What we will do next assumes constant coefficients.
- Textbook does not emphasize the transition
The method of undetermined coefficients

This method applies to a second-order linear equation with constant coefficients if the right-hand side $F(t)$ has one of a few particularly simple forms:

- A polynomial
- An exponential times a polynomial: $e^{\alpha t} P(t)$. The exponent is a constant times $t$.
- Complex exponentials are allowed, so we also can handle $P(t) \cos t$,
- $P(t) \sin t$
- $e^{\alpha t} \cos t$, etc.
Also the right-hand side can be a linear combination of expressions of those forms, since if

\[ au'' + bu' + cu = F(t) \]

and

\[ av'' + bv' + cv = G(t) \]

then \( y = au + bv \) satisfies

\[ ay'' + by' + cy = aF(t) + bG(t) \]
The method of undetermined coefficients

- Assume that the sought-after solution $y$ has the same form as the right-hand side $P(t)$.
- That assumption mentions some unknown coefficients.
- Plug in that solution to the differential equation and simplify.
- You get equations for the coefficients.
- If they are solvable you are done.
Example: $y'' - 3y' - 4y = 3e^{2t}$

- $3e^{2t}$ is (a constant times) an exponential.
- Assume $y = Ae^{2t}$.
- Plug and grind:

\[
\begin{align*}
y' &= 2Ae^{2t} \\
y'' &= 4Ae^{2t} \\
y'' - 3y' - 4y &= 4Ae^{2t} - 3(2Ae^{2t} - 4Ae^{2t}) \\
&= -6Ae^{2t} \\
&= 3e^{2t} \quad \text{if this is to be a solution} \\
A &= -\frac{1}{2} \quad \text{the equation is solvable, so it works.}
\end{align*}
\]
Example: \( y'' - 3y' - 4y = 3e^{2t} \)

- We found the particular solution
  \[
y = -\frac{1}{2}e^{2t}
\]

- To find the general solution we need to solve the homogeneous equation too.
- The characteristic equation is
  \[
  \lambda^2 - 3\lambda - 4 = 0
  \]
  which has roots \( \lambda = 4 \) and \( \lambda = -1 \).
- So the general solution of the homogeneous equation is
  \[
ae^{4t} + be^{-t}
  \]
- and the general solution of the non-homogeneous equation is
  \[
ae^{4t} + be^{-t} - \frac{1}{2}e^{2t}.
  \]
Special cases can cause trouble

- If the proposed solution of the non-homogeneous equation is actually already a solution of the homogeneous equation, then the equations for the coefficients cannot be solved.

- For example, in the preceding problem, the homogeneous equation had solutions $e^{-t}$ and $e^{4t}$. What if the right-hand side had been $e^{4t}$?

$$y'' - 3y' - 4y = e^{4t}$$

- Then we would have assumed $y = Ae^{4t}$, but when we plug it in, we get 0 on the left, which can never be equal to $e^{4t}$ on the right.

- So the method needs modification in such cases.
What to do in a special case

- If the proposed solution of the non-homogeneous equation is already a solution of the homogeneous equation, then the assumed form should be multiplied by a factor of $t$.

- For example:

  $y'' - 3y' - 4y = e^{4t}$

  Since $e^{4t}$ is a solution of the homogeneous equation, we instead assume

  $$y = Ate^{4t}.$$ 

- Now we plug and grind:

  $$y' = Ae^{4t}(4t + 1)$$

  $$y'' = Ae^{4t}(4 + 4(4t + 1)) = Ae^{4t}(16t + 8)$$

  $$y'' - 3y' - 4y = Ae^{4t}(16t + 8) - 3Ae^{4t}(4t + 1) - 4Ate^{4t}$$

  $$= -2Ae^{4t}$$

- So if we take $A = -1/2$ we have a solution.

- Note that the $te^{4t}$ terms canceled out. That’s because $e^{4t}$ solves the homogeneous equation.
Another special case

- If the right-hand side is already a solution of the homogeneous equation, and
- if in addition the characteristic equation has double roots, then
- multiply by $t^2$ instead of only $t$.
- For example

$$y'' + 2y' + 1 = e^{-t}$$

- The characteristic equation is $(\lambda + 1)^2 = 0$, so the homogeneous equation has solutions $e^{-t}$ and $te^{-t}$. So the right side is a solution of the homogeneous equation, but so is $te^{-t}$ (which we would otherwise try as a solution). So instead we try

$$y = At^2 e^{-t}$$

and you can check that it works.
Summary of the Method of Undetermined Coefficients

- It’s for linear non-homogeneous second-order equations with constant coefficients.
- Assume a solution that has the same form as the right hand side.
- That is, a polynomial, or an exponential or trig function times a polynomial.
- Use letters for the polynomial coefficients and solve for them.
- Use an extra factor of $t$ if the right side already solves the homogeneous equation, or an extra factor of $t^2$ if in addition the characteristic equation has multiple roots.
- For proof that it works, see pages 181-182 of the text. You just use letters for the coefficients of the right-hand side and plug and grind as you do when solving a particular example.
Variation of Parameters

- This method “works” on any second-order non-homogeneous equation, constant coefficients or not.
- But the “solution” involves an integral, so it may be harder to work with.
- Also it requires have a fundamental set of solutions of the homogeneous equation, which may not be easy if the equation doesn’t have constant coefficients.
- Therefore use the method of undetermined coefficients if it is applicable.
Variation of Parameters

We consider the equation

\[ y'' + p(t)y' + q(t)y = g(t) \]

and suppose we have somehow found a fundamental set of two solutions \( y_1 \) and \( y_2 \) of the homogeneous equation

\[ y'' + p(t)y' + q(t)y = 0. \]

The basic idea is to look for a solution in the form

\[ y = uy_1 + vy_2 \]

where \( u \) and \( v \) are not constants, but functions of \( t \).
Variation of Parameters

Our plan is to plug

\[ y = uy_1 + vy_2 \]

into the equation

\[ y'' + p(t)y' + q(t)y = g(t) \]

So we start by differentiating \( y \):

\[ y' = u'y_1 + u'y_1' + v'y_2 + v'y_2 \]

Now we assume \( u'y_1 + v'y_2 = 0 \). Then

\[ y' = uy'_1 + vy'_2 \]

\[ y'' = u'y_1 + uy''_1 + v'y_2 + v''_2 \]
So now plug the expressions for $y'$ and $y''$ into the original equation. Specifically, plug

$$y' = uy'_1 + vy'_2$$
$$y'' = u'y_1 + uy''_1 + v'y_2 + v''_2$$

into

$$y'' + p(t)y' + q(t)y = g(t)$$

The coefficient of $u$ is $y''_1 + py'_1 + qy_1$, which is zero since $y_1$ is a solution of the homogeneous equation. The coefficient of $v$ is similarly zero since $y_2$ is a solution. We are left with

$$u'y'_1 + v'y'_2 = g(t).$$
We have proved that if we solve the equations

\[ u' y_1 + v' y_2 = 0 \quad \text{which we assumed above} \]
\[ u'y_1' + v'y_2' = g(t) \]

then \( y = uy_1 + vy_2 \) will solve the non-homogeneous equation.

But these are algebraic equations for \( u' \) and \( v' \).

The solution has the Wronskian \( W(y_1, y_2) \) in the denominator, which is nonzero.

So it’s always possible to solve for \( u' \) and \( v' \).

Then if we can integrate the answers, we have our solution.
Solving
\[ u' y_1 + v' y_2 = 0 \]
\[ u' y_1' + v' y_2' = g(t) \]
we get
\[ u' = -\frac{y_2 g}{W} \]
\[ v' = \frac{y_1 g}{W} \]
where \( W \) is the Wronskian
\[ W = y_1 y_2' - y_2 y_1' \]
Therefore a solution is
\[ u = -\int \frac{y_2(t)g(t)}{W(t)} dt + c_1 \]
\[ v = \int \frac{y_1(t)g(t)}{W(t)} dt + c_2 \]
\[ y = uy_1 + vy_2 \]
An example: \( y'' + 4y = 3 \csc t \)

- Although the coefficients are constant, the right side is not a polynomial times an exponential.
- So we can’t use the method of undetermined coefficients.
- We can solve the homogeneous equation, since the coefficients are constant.
- The details of this example are on pages 185-187, presented as a motivation for the method of variation of parameters.
An example: \( y'' - 3y' - 4y = t^2 \)

- What method should we use?

Undetermined coefficients, since we have a polynomial on the right.

- What form should we assume for the solution?

\( y = a + bt + ct^2 \). No exponential, since the right side is a polynomial.

Plugging this into the equation we find it will work if

\[
2c - 3b - 4a = 0
\]

\[
-6c - 4b = 0
\]

\[
-4c = 1
\]

It is possible to solve these equations:

\[
c = -\frac{1}{4}, \quad b = \frac{3}{8}, \quad a = -\frac{13}{32}
\]

- What is the general solution?

\( y = -\frac{13}{32} + \frac{3}{8}t - t^2 + Ae^{-t} + Be^{4t} \).
An example: $y'' - 3y' - 4y = t^2$

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- It is possible to solve these equations:
  - \( c = -1/4, b = 3/8, a = -13/32 \)
- What is the general solution?
  \[
  y = -13/32 + 3/8t - t^2 + Ae^{-t} + Be^{4t}.
  \]
An example: $y'' - 3y' - 4y = t^2$

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  $c = -1/4, b = 3/8, a = -13/32$
- What is the general solution?
  
  \[
y = -\frac{13}{32} + \frac{3}{8}t - \frac{t^2}{4} + Ae^{-t} + Be^{4t}.
  \]