A Conjecture on Edge–Graceful Trees

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Dedicated to Professor Roberto W. Frucht on the occasion of his 80th birthday.

Abstract. A graph \((V,E)\) is said to be edge–graceful if there exists a bijection \(f : E \rightarrow \{1,2,\ldots,|E|\}\) such that the induced mapping \(f^+ : V \rightarrow \{0,1,\ldots,|V| - 1\}\) defined by

\[
f^+(v) = \sum_{(u,v) \in E} f(u,v) \pmod{|V|}
\]

is a bijection. We conjecture that every tree with an odd number of vertices is edge–graceful. We show that every odd tree with at most one node with degree 2 is edge–graceful. Several classes of trees for which the above conjecture holds are given.

1. Introduction

In this paper, a graph \(G = (V,E)\) is a finite undirected graph without multiple edges and loops. Let \(G = (V,E)\) with \(|V| = p\) and \(|E| = q\). The graph \(G\) is said to be edge–graceful if there exists a bijection \(f : E \rightarrow \{1,2,\ldots,q\}\) such that the induced mapping \(f^+ : V \rightarrow \{0,1,\ldots,p - 1\}\), which is defined by \(f^+(v) = \sum f(e) \pmod{p}\) for all \(e\) in \(E(G)\) and incident to \(v\), is a bijection.

Lo [7] initiated the study of edge–graceful graphs. He showed that a tree of \(2k\) vertices is not edge–graceful. In [7] the stars and paths with odd number of nodes are shown to be edge–graceful.

We propose the following conjecture.

Conjecture. Every tree with an odd number of vertices is edge–graceful.

A graph \(G = (V,E)\) is said to be graceful if there exists a 1–1 mapping \(f : V \rightarrow \{0,1,\ldots,|E|\}\) such that the induced mapping \(f^* : E \rightarrow \{1,2,\ldots,|E|\}\) which is defined by \(f^*((a,b)) = |f(a) - f(b)|\) for all edges \((a,b)\) in \(E\) is a bijection (see [3]).

In my opinion, the proof of the above conjecture is as hard as the following famous conjecture.

Ringel–Kotzig Conjecture [1]. Every tree is graceful.

Much effort has been extended towards proving this conjecture. A number of papers have appeared in the past few years which deal with various graceful labelings of trees (e.g., see [2], [5], [8]). In this paper we try to give some partial results of our conjecture by showing some classes of tree which are edge–graceful.