4. Proof of the Theorems.

The graph \( SP(a_1, \ldots, a_t) \) consists of \( t \) partites, where each partite is a null graph \( K_{a_i} \), \( i = 1, \ldots, t \). We label \( SP(a_1, \ldots, a_t) \) according to the first algorithm.

We observe that the vertices in odd columns from bottom to top and from left to right are labeled with integers \( 0, 1, 2, \ldots, \lfloor t/2 \rfloor \)
\( \sum_{i=1}^{\lfloor t/2 \rfloor} a_{2i-1} \) - 1. The vertices in even columns have different labeling which range between \( \sum_{i=1}^{\lfloor t/2 \rfloor} a_{2i-1} \) and \( \sum_{i=1}^{t-1} a_{a_i+1} \). It is clear that the vertex labeling \( f \) defined by the algorithm is graceful. This prove Theorem 1.

Denote \( r = ( \sum_{i=1}^{\lfloor t/2 \rfloor} a_{2i-1} ) - 1 \). Then we see that the labeling \( f \) is an \( \alpha \)-valuation, with \( A = \{ u : f(u) > r \} \) and \( B = \{ u : f(u) \leq r \} \).

We see that the bipartition \( B = (u_1, j : i \text{ is odd, } 1 \leq j \leq a_i) \) and
\( A = (u_i, j : i \text{ is even, } 1 \leq j \leq a_i) \) has the property that \( f(a) > f(b) \) for all \( a \in A \) and \( b \in B \). As a consequence of a result of Maheo and Thuillier [16] we have for any \( k \geq 2 \), the function \( f_k \) defined on the set \( V(SP(a_1, \ldots, a_t)) \) by setting
\[
f_k(u) = \begin{cases} 
 f(u) + k - 1 & \text{if } u \in A \\
 f(u) & \text{if } u \in B 
\end{cases}
\]

Then the mapping \( f_k \) is a \( k \)-graceful labeling.

The analogous verification that the vertex labeling given by the 2nd algorithm is an \( \alpha \)-valuation will be omitted.

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