Directions and rules. The exam will last 70 minutes; the last five minutes of class will be used for collecting the exams. No electronic devices of any kind will be allowed, with one exception: a music player that nobody else can hear, and whose controls you do not use during the exam (just put it on shuffle). Anything (else) with an off switch must be off. In particular, turn your cell phone off. If it says the answer must be a number, it means an integer or a decimal number; thus 5/11 or 2π should be evaluated further.

Show your work. There are 7 problems, total 100 points. Five of the problems are worth 14 points each, and the two problems with three parts are worth 15 points each (five points per part). You might get some partial credit, but only if you have done part of the problem completely correctly, not if you’ve written some lines that could perhaps be edited to be correct.

1. As a consulting scientist, you have been given a sphere of zirconium and asked to determine its volume by measuring the diameter. The diameter is approximately 2 centimeters. Your client wants to know the volume to the nearest 0.001 cubic centimeter. How accurately must you measure the diameter to achieve the desired accuracy in volume? In other words, you must measure the diameter accurate to within approximately how many centimeters? Use differentials to solve this problem. The answer must be a number.

Solution:

\[
V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \frac{D^3}{8} = \frac{1}{6}\pi D^3
\]

\[
dV = \frac{\pi}{2}D^2 dD
\]

\[
dD = \frac{2dV}{\pi D^2} = \frac{2 \cdot 0.001}{\pi \cdot 2^2} = \frac{0.001}{6.28}
\]

= 0.00016 cm.
2. The volcano at Thera, Greece (also called Santorini) erupted many centuries BCE (before Christian era). That eruption completely destroyed the island of Santorini and that event may be the basis for the story of the “lost continent” Atlantis that supposedly sank into the sea. When did this eruption occur? It was discovered in 2006 that there was an olive tree buried under the lava flow. Suppose that measurements (in 2006) revealed that the carbon-14 content of the remains of the olive tree was 32.33% of what it had been at the time of the eruption. (The true number was different, so you won’t get the true historical date when you solve this problem.) Using 5730 years for the half life of carbon 14, calculate the date of the eruption. The answer must be an integer number for the date (not for the number of years that have elapsed since that date).

To do this problem without a calculator you can make use of some of these values:

\[
\begin{align*}
\log_2(1/0.3233) & = -\log_2(0.3233) = 1.63 \\
\ln 2 & = -\ln(1/2) = 0.693 \\
\ln 0.3233 & = -1.129
\end{align*}
\]

**Solution.**

\[
\begin{align*}
(1/2)^{t/5730} & = 0.3233 \\
1 & = 0.3233 \cdot 2^{t/5730} \\
\log_2(1/0.3233) & = t/5730 \\
1.63 & = t/5730 \\
t & = 5730 \cdot 1.63 = 9340
\end{align*}
\]

So the answer is 9340 – 2006 = 7334 BCE.

Alternately 0.3233 = \( e^{-ct} \) where \( e^{-5730c} = 1/2 \); so then \( c = \ln(2)/5730 = 0.693/5730 = 0.000121 \). Then \( -ct = \ln 0.3233 \) so \( t = -\ln(0.3233)/c = 1.129/0.000121 = 1129000/121 = 9330 \) years ago, which corresponds to 7324 BCE, close enough to the answer we got by the first method—they differ by only one part in a thousand, and we used some three-digit approximate numbers.
3. A plane flying horizontally with a constant speed of 480 km/hr passes over a ground radar station at an altitude of 6 km. At what rate is the distance from the plane to the radar station increasing one minute later? *The answer must be a number and the units of that number.*

**Solution:**

\[
s^2 = x^2 + 6^2
\]

\[\frac{dx}{dt} = 8 \text{ km/min}\]

After one minute \(x = 8\), so \(s = 10\) (either because you see that the triangle has the shape 3-4-5, or by calculating \(s = \sqrt{6^2 + 8^2}\)). Now differentiate the main equation:

\[
s^2 = x^2 + 6^2
\]

\[2s \frac{ds}{dt} = 2x \frac{dx}{dt}
\]

\[s \frac{ds}{dt} = x \frac{dx}{dt} = 8\text{ since } \frac{dx}{dt} = 8
\]

At the particular moment of interest we have \(x = 8\) and \(s = 10\):

\[
10 \frac{ds}{dt} = 8 \cdot 8 = 64
\]

\[
\frac{ds}{dt} = \frac{64}{10} = 6.4 \text{ km/min} = 384 \text{ km/hr}
\]

4. (a) sketch the graph of \(\cosh x\)

(b) sketch the graph of \(\sinh x\)

(c) sketch the graph of \(\tanh x\)

You can look these graphs up in the textbook or draw them in MathXpert.

5. Differentiate each of the following with respect to \(x\):

(a) \(\cosh(1/x)\)

\[
\frac{d}{dx} \cosh(1/x) = \sinh(1/x) \frac{d}{dx}(1/x)
\]

\[= \sinh(1/x) \left( -\frac{1}{x^2} \right)
\]

\[= -\frac{\sinh(1/x)}{x^2}
\]
(b) \( \tanh(\sqrt{x}) \)

\[
\frac{d}{dx} \tanh(\sqrt{x}) = \operatorname{sech}^2(\sqrt{x}) \frac{d}{dx} (\sqrt{x})
\]

\[
= \operatorname{sech}^2(\sqrt{x}) \left( \frac{1}{2\sqrt{x}} \right)
\]

\[
= \frac{\operatorname{sech}^2(\sqrt{x})}{2\sqrt{x}}
\]

You could also write this as

\[
\frac{1}{2\sqrt{x} \cosh^2 x}
\]

6. Use the definitions of \( \sinh \) and \( \cosh \) to derive the formula \( \cosh(2x) = 1 + 2 \sinh^2 x \). Show your work.

\[
\cosh(2x) =? 1 + 2 \sinh^2 x
\]

\[
e^{2x} + e^{-2x} =? 1 + 2 \left( \frac{e^x - e^{-x}}{2} \right)^2
\]

\[
e^{2x} + e^{-2x} =? 2 + 4 \left( \frac{e^x - e^{-x}}{2} \right)^2
\]

\[
e^{2x} + e^{-2x} =? 2 + (e^x - e^{-x})^2
\]

\[
= 2 + e^{2x} - 2 + e^{-2x}
\]

\[
= e^{2x} + e^{-2x}
\]

Since the two sides are now identical, the identity is verified.

Here are some hints for doing calculations with pencil and paper instead of a calculator. The sign \( \cong \) means “approximately equal to.” For small numbers \( x \), we have \( \ln(1 + x) \cong x \), and we have \( e^x \cong 1 + x \), or even more accurately, \( e^x \cong 1 + x + x^2/2 \). So for example, \( \ln \frac{101}{100} \cong \ln 1.01 \cong 0.01 \). A useful number to know is \( \ln 10 \cong 2.303 \). Then for example \( \ln 11 \) can be found as \( \ln(10 \cdot 1.1) = \ln 10 + \ln 1.1 \cong 2.3 + 0.1 = 2.4 \). With these hints, you won’t need a calculator on the following problem:

7. A cup of coffee is served at 180° F. The room temperature is 70°. After one minute the coffee has cooled to 170°. Use Newton’s Law of Cooling to answer these questions.

(a) What equation does Newton’s Law of Cooling say should describe the temperature of the coffee?
\[ T - 70 = (180 - 70)e^{-ct} = 110e^{-ct} \]

where \( t \) is time since the coffee was served, and \( c \) is a constant. We have to find \( c \). When \( t = 1 \) we have \( T = 170 \), so \( 100 = 110e^{-c} \). Hence \( c = -\ln \frac{100}{110} = \ln \frac{110}{100} = \ln 1.1 \approx 0.1 \).

(b) What will the temperature of the coffee be after one more minute (that is, two minutes after it is served)? \( \text{The answer must be a number.} \)

\[
T = 70 + 110e^{-ct}
= 70 + 110e^{-2\ln 1.1} \text{since } c = \ln 1.1 \text{ and } t = 2
= 70 + 110(\ln 1.1)^{-2}
= 70 + 110 \cdot 1.1^{-2}
= 70 + \frac{110}{1.1^2}
= 70 + \frac{110}{1.21}
= 70 + 90.9 = 161^\circ
\]

That seems reasonable: ten degrees cooler in the first minute, another nine degrees cooler in the second minute.

(c) When will the coffee cool to 80\(^{\circ}\)? That is, how many minutes after being served? \( \text{The answer must be a number.} \)

\[
80 - 70 = 110e^{-ct}
\frac{10}{10} = 110e^{-(\ln 1.1)t}
\frac{110}{110} = e^{-(\ln 1.1)t}
\ln \frac{11}{11} = -t \ln 1.1
-t \ln 1.1 = \frac{\ln 11}{\ln 1.1}
-t = \frac{2.4}{0.1} \text{ see the hints}
= 24
\]

That’s the answer, 24 minutes after serving.