MIDTERM EXAM 3

Directions and rules. The exam will last 70 minutes; the last five minutes of class will be used for collecting the exams. No electronic devices of any kind will be allowed, with one exception: a music player that nobody else can hear, and whose controls you do not use during the exam (just put it on shuffle). Anything (else) with an off switch must be off. In particular, turn your cell phone off.

Show your work. Scoring: 6 problems, each worth 16 points, plus 4 free points = 100.

1. “Eve of Naharon” is the skeleton of a 25 to 30 years old female found in the Naharon section of the underwater cave Sistema Naranjal in Mexico near the town of Tulum, around 80 miles south west of Cancún. Radiocarbon analysis showed that 19.3% of the original amount of carbon 14 remained. How old is the skeleton?

To do this problem without a calculator you can make use of some of these values:

\[
\begin{align*}
\ln(0.193) & = -1.64506509 \\
\ln 2 & = -\ln(1/2) = 0.693 \\
\log_2(0.193) & = -2.37332725
\end{align*}
\]

Solution:

\[
\begin{align*}
(1/2)^{t/5730} & = 0.193 \\
2^{-t/5730} & = 0.193 \\
-t/5730 & = \log_2(0.193) = -2.37333 \\
t & = 5730 \cdot 2.3733 \\
& = 13,600
\end{align*}
\]

Date: November 21, 2011.
2. Consider a cylindrical storage tank, ten meters high. Use differentials to determine how accurately we must measure the interior diameter of the tank, in order to calculate the tank’s volume to within 1% of its true value? (Give your answer in percent.)

Solution: Let \( V \) be the volume of the tank and \( r \) the radius. Then

\[
\begin{align*}
V &= 10\pi r^2 \\
dV &= 20\pi r dr \\
\frac{dV}{V} &= \frac{2dr}{r} \\
\frac{dV}{V} &= 0.01 \\
\frac{dr}{r} &= \frac{1}{2C} \frac{dV}{V} = 0.005
\end{align*}
\]

So the radius must be measured with a relative error of at most 0.005, which is 0.5%. The percentage error in the diameter is the same as the percentage error in the radius, 0.5%.

3. (a) sketch the graph of \( \cosh x \)

(b) sketch the graph of \( \sinh x \)

(c) sketch the graph of \( \tanh x \)

These graphs can be found in the textbook or by using MathXpert; therefore it isn’t worth the trouble to create electronic versions of them to paste in here.

4. Differentiate each of the following with respect to \( x \), expressing the answer using hyperbolic trig functions (i.e., not using \( e^x \)).

(a) \( \cosh(\sqrt{x}) \)

\[
\begin{align*}
\frac{d}{dx} \cosh(\sqrt{x}) &= \sinh(\sqrt{x}) \frac{d}{dx} \sqrt{x} \\
&= \sinh(\sqrt{x}) \frac{1}{2\sqrt{x}} \\
&= \frac{\sinh \sqrt{x}}{2\sqrt{x}}
\end{align*}
\]

(b) \( \tanh(1/x) \)

\[
\begin{align*}
\frac{d}{dx} \tanh(1/x) &= \operatorname{sech}^2(1/x) \frac{d}{dx} (1/x) \\
&= \operatorname{sech}^2(1/x) \frac{-1}{x^2} \\
&= -\frac{\operatorname{sech}^2(1/x)}{x^2}
\end{align*}
\]
5. Use the definitions of sinh and cosh to verify the identity.

\[ \cosh 2x = \cosh^2 x + \sinh^2 x \]

\[
\frac{\cosh 2x}{e^{2x} + e^{-2x}} \quad =? \quad \frac{\cosh^2 x + \sinh^2 x}{2} \\
=\frac{\left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2}{2} \\
=\frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} + \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4} \\
=\frac{2e^{2x} + 2e^{-2x}}{4} \\
=\frac{e^{2x} + e^{-2x}}{2}
\]

Since both sides are now the same, the identity is verified.

6. A girl flies a kite at a height of 300 feet. The wind carries the kite horizontally away from her at a rate of 25 ft/sec. How fast must she let out more string when she has already let out 500 feet of string? Assume that the string forms a straight line from the girl to the kite.

(a) Introduce two variables (in addition to \( t \) for time) and draw a picture showing what the variables mean.

```
\begin{center}
\begin{tikzpicture}[scale=0.7]
\filldraw[black] (0,0) circle (2pt) node[anchor=north west] {girl} -- (2,4) node[anchor=west] {kite} -- (4,0) node[anchor=north west] {300} -- cycle;
\end{tikzpicture}
\end{center}
```

(b) Write an equation or equations connecting your variables, true for all the time the kite is flying as described.

\[ x^2 + 300^2 = z^2 , \quad \text{and} \quad dx/dt = 25 \]

(c) Write one or more additional equations that are only good at one special time.

\[ z = 500; \quad \text{so} \quad x = 400 \quad \text{at that time} \]

(d) Finish solving the problem, to get the answer in feet per second.
Differentiating the equation in part (b) we have

$$2x \frac{dx}{dt} = 2z \frac{dz}{dt}.$$ 

Since \(dx/dt = 25\) we have \(dz/dt = 25x/z = 25 \cdot 4/5 = 20\) ft/sec.