SOLUTIONS TO ASSIGNMENT 18 FOR MATH 30

1. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.
   (a) Find a formula for the area inside the ripple at time $t$.
   
   $$ A = \pi (60t)^2 = 3600\pi t^2 $$

   (b) Use calculus to find the rate at which the area is increasing at time $t$.
   
   $$ \frac{dA}{dt} = 7200\pi t $$

   Alternately we could do this:
   
   $$ \frac{dA}{dt} = \frac{d}{dt} \pi r^2 = 2\pi r \frac{dr}{dt} = 2\pi 60t \cdot 60 $$

   $$ = \frac{7200\pi t}{7200\pi t} $$

   (c) Find the numerical rate at which the area is increasing 5 seconds after the stone is dropped.

   Putting $t = 5$ into $7200\pi t$ we get $36000\pi = 113097$ cm$^2$ per second. (Many students stopped with $36000\pi$, and did not write the units.)

2. Boyle’s Law says that when a sample of gas is compressed at a constant temperature, the product of the press and the volume remains constant: $PV = C$. Find the rate of change of volume with respect to pressure.

   $$ V = \frac{C}{P} $$
   $$ \frac{dV}{dP} = -\frac{C}{P^2} $$

   by the reciprocal rule. $C$ is a constant, so there’s no need to work on $dC/dP$.

3. The quantity of insulin in a diabetic patient’s bloodstream after an injection decreased according to the equation $I = 2 - (0.02)t^2$, where $t$ is measured in hours, and $I$ in milliliters. Find the rate of change of the amount of insulin one hour after the injection. (Of course this equation is only good for a few hours, not forever, but that’s enough for this problem.)

   $$ \frac{dI}{dt} = -2(0.02t) = -0.04t $$
Now put $t = 1$, and we get $-0.04$ milliliters per hour. (Many students differentiated $I$ to get $dI/dt = 2 - 0.04t$. I am not sure why they did that! And almost nobody wrote down the units of $dI/dt$.)

4. If $f$ is the focal length of a convex lens and an object is placed at a distance $p$ from the lens, then its image will be at a distance $q$ from the lens, where $f$, $p$, and $q$ are related by the *lens equation*

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}.$$  

For example, a photographer might use a 28 mm. lens or a 50 mm. lens; here the 28 mm or 50 mm is the focal length of the lens.

(a) Draw a picture of the lens and an object and label the distances $p$ and $q$.

*Solution.* It’s too hard to get such a picture into a pdf file. If you will google for “$1/f = 1/p + 1/q$ focal length” you’ll find a lot of information about the applications of this equation, and a lot of pictures, but I didn’t find the simple one I had in mind, which I’ll draw on the board in class.

(b) Find the rate of change of $p$ with respect to $q$. *Solution.* We start by solving for $p$, then differentiate, remembering the $f$ is a constant (it only depends on the lens, which is fixed for this problem), so $df/dq = 0$.

$$\frac{1}{p} = \frac{1}{f} - \frac{1}{q}$$

$$-\frac{dp/dq}{p^2} = \frac{1}{q^2}$$

$$\frac{dp}{dq} = \frac{-p^2}{q^2}$$

I accepted that answer, but we could also express the answer in terms of $q$ alone, by expressing $p$ in terms of $q$. You can also first solve explicitly for $p$ and then differentiate that expression.

(c) Why would a camera designer want to know about this rate of change?

*Solution:* It controls how far towards or away from the camera the subject can move before going out of focus.

5. Let $y(t)$ be the population of planet Earth in year $t$. What is the (approximate) value of $dy/dt$ at the present time? You will need to look up some numbers on the Internet to answer this question, as well as understand what $dy/dt$ means in this case.

*Hints.* You do not need and *should not try to find* a formula for the population as a function of time. If you do not see how to proceed, think about the definition of the derivative, and apply it to this case.

*Solution.* According to the definition of derivative, we can approximate $dy/dt$ by

Well, I couldn’t find those numbers immediately, but I found

\[ y(2011) = 7 \text{ billion} \]
\[ y(2009) = 6.8 \text{ billion} \]

which makes \( dy/dt \) approximately \((y(2011) - y(2009))/2 = 100,000,000\) people per year.

Different numbers can also be found on the Web, it seems. We can’t really know what the population is today or last month with great accuracy so we have to settle for these approximations. Note that this one really is an approximation to \( dy/dt \) in 2010, not today, but it’s probably not much different.

6. At a place called Roopkund in the Indian Himalayas, more than five hundred human skeletons were found. Medical experts say they died by being caught without shelter in a hailstorm with large hailstones. Although these bones were known for decades, they were only radiocarbon dated in 2004. Assume the half-life of carbon 14 is 5730 years. The researchers found that 87% of the original carbon 14 was still present. What was the date of the fatal hailstorm? Here are some useful numbers: \( \log_2(0.87) = -0.2009; \ln(2) = 0.693; \ln(0.87) = -0.1393. \)

Please do the arithmetic required to get an actual date. The only acceptable answer is an integer for the year of the hailstorm. You only need to be accurate within ten years—radiocarbon dating is not more accurate than that anyway.

**Solution.** Let \( t \) be the number of years the skeleton was decaying before 2004.

\[
0.87 = \left( \frac{1}{2} \right)^{t/5730} = 2^{-t/5730} \\
\log_2 0.87 = -\frac{t}{5730} \\
t = -5730 \log_2 0.87 \\
= -5730(-0.2009) \\
= 1151 \text{ years ago}
\]

More than half the students stopped with “1151”, without putting any units on it, and apparently not realizing what it means. It is the number of years before 2004 since the hailstorm. So, to find the date, we subtract 1151 from 2004, yielding the desired date of 853 CE. The problem says *The only acceptable answer is an integer for the year of the hailstorm.* On the homework I accepted 1151, but on the exam, you will have to live with *The only acceptable answer is an integer for the year of the event being dated.*

It’s also possible to solve the problem using \( e \) instead of \( 1/2 \) as the base of the exponential:

\[
0.87 = e^{ct} \\
0.5 = e^{5730c}
\]
Then you have to solve the second equation for \( c \), put it into the first equation, and solve for \( t \):

\[
\ln 0.5 = 5730c \\
c = \frac{\ln 0.5}{5730} = -\frac{\ln 2}{5730} \\
0.87 = e^{-\left(t \ln 2\right)/5730} \\
\ln 0.87 = -\frac{t \ln 2}{5730} \\
t = -\frac{5730 \ln 0.87}{\ln 2} \\
= -\frac{5730 \cdot (-0.1393)}{0.693} \\
= -\frac{5730 \cdot 0.1393}{0.693} \\
= 1152 \text{ years ago.}
\]

To do this without a calculator let’s first get the answer to one significant digit. Then it’s about \( 6000 \cdot (1/7)/0.7 \) since 0.14 is about 1/7. So that’s about \( 6000/4.9 \) which is about 1200. Since we need ten-year accuracy, we better keep three digits, so there’s some arithmetic to do. Note that in the first solution, we didn’t need to do much arithmetic, and there was no constant \( c \) to find.

7. A bacterial culture starts with 500 bacteria and after 3 hours there are 8000 bacteria.

(a) Find an expression for the number of bacteria after \( t \) hours.

\textit{Solution.} In three hours the population was multiplied by 16. So

\[ N = 500 \cdot 16^{t/3} \]

That’s a good answer to part (a).

Alternately, we can use \( e \) as the base of the exponential, and then

\[ N = 500e^{ct} \quad \text{and we have to find } c. \]

\[ 8000 = 500e^{3c} \]

\[ 16 = e^{3c} \]

\[ \ln 16 = 3c \]

\[ c = \frac{\ln 16}{3} = 0.924 \]

\[ N = 500e^{0.924t} \quad \text{That’s another answer to part (a)} \]

(b) Find the number of bacteria after 4 hours

\[ N = 500 \cdot 16^{4/3} = 20,159 \]

(c) When will the population reach 30,000?
\[ 30,000 = 500 \cdot 16^{t/3} \]
\[ 60 = 16^{t/3} \]
\[ = e^{(t/3) \ln 16} \]
\[ \ln 60 = (t/3) \ln 16 \]
\[ t = \frac{3 \ln 60}{\ln 16} \]
\[ = 4.43 \text{ hours} \]

8. After 3 days a sample of radon-222 decayed to 58% of its original amount.
(a) What is the half-life of radon-222?

Solution. Let \( t \) be the half life. Then
\[ 0.5 = e^{ct} \]
\[ 0.58 = e^{3c} \]
\[ \ln 0.58 = 3c \]
\[ c = \frac{\ln 0.58}{3} \]
\[ = -0.1816 \quad \text{It better be negative!} \]
\[ 0.5 = e^{-0.1816t} \quad \text{from the first equation, now that we know} \ c \]
\[ \ln 0.5 = -0.1816t \]
\[ t = \frac{\ln 0.5}{-0.1816} \]
\[ = 3.817 \text{ days} \]

Note: The answer better be positive, and it better be more than 3 days, since after three days there was still 58% left, so half of it had not decayed yet. Many students turned in answers less than three days.

(b) How long would it take the sample to decay to 10% of its original amount?
\[ 0.1 = e^{ct} = e^{-0.1816t} \]
\[ \ln 0.1 = -0.1816t \]
\[ t = \frac{\ln 0.1}{-0.1816} \]
\[ = 12.68 \text{ days} \]
9. A thermometer is taken from a room where the temperature is 20° C to the outdoors where the temperature is 5° C. After one minute the thermometer reads 12°. Use Newton’s Law of Cooling to answer these questions:

(a) What will the reading on the thermometer be after one more minute?

\[ T - 5 = 15e^{ct} \]

To find \( c \) we use the second sentence of the problem:

\[
\begin{align*}
12 - 5 &= 15e^c \\
7 &= 15e^c \\
\ln 7 &= \ln 15 + c \\
c &= \ln 7 - \ln 15 = -0.762
\end{align*}
\]

Now after one more minute, \( t = 2 \), so

\[
\begin{align*}
T &= 5 + 15e^{-0.762 \cdot 2} \\
&= 5 + 15e^{-1.524} \\
&= 8.27°
\end{align*}
\]

Many students forgot to write the units. The answer has to be colder than 12° since it was at that temperature earlier. It has to be warmer than the outside temperature of 5°. Some students turned in answers that violated these common-sense observations.

(b) When will the thermometer read 6° C?

\[
\begin{align*}
6 - 5 &= 15e^{-0.762 \cdot t} \\
1 &= 15e^{-0.762 \cdot t} \\
\ln 1 &= 0 = \ln 15 - 0.762t \\
t &= \frac{\ln 15}{0.762} \\
&= 3.55 \text{ minutes}
\end{align*}
\]

Many students forgot to write the units. The answer better be more than two minutes, since after two minutes it was still warmer than 8 degrees.