More NP-complete Problems

CS255

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Outline

- More NP-Complete Problems

Hamiltonian Cycle

- Recall a **hamiltonian cycle** is a permutation of the vertices $v_{i\_1},\ldots, v_{i\_n}$ of a graph G so that there is an edge between $\{v_{i\_j}, v_{i\_j+1}\}$ for each j as well an edge $\{v_{i\_n}, v_{i\_1}\}$.
- Let *HAM-CYCLE* be the language $\{<G> \mid G$ contains a hamiltonian cycle$\}$.

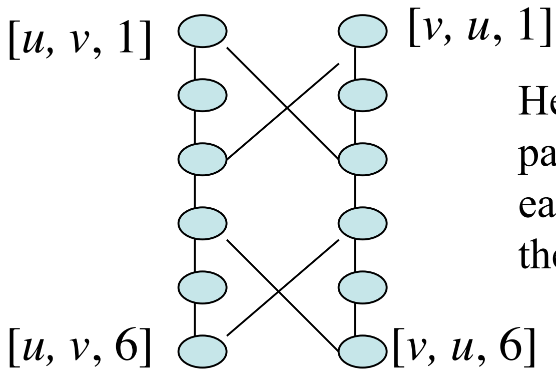
Theorem. *HAM-CYCLE* is **NP**-complete.

Proof. First, given a permutation of the vertices, we can in polynomial time verify whether or not it is a hamiltonian cycle. So HAM-CYCLE is in **NP**. To see it is **NP**-complete, we show VERTEX-COVER$\leq_p$ HAM-CYCLE. Given a graph G and an integer k, we need to make a new graph G′which has a hamiltonian cycle iff the original had a vertex cover of size.

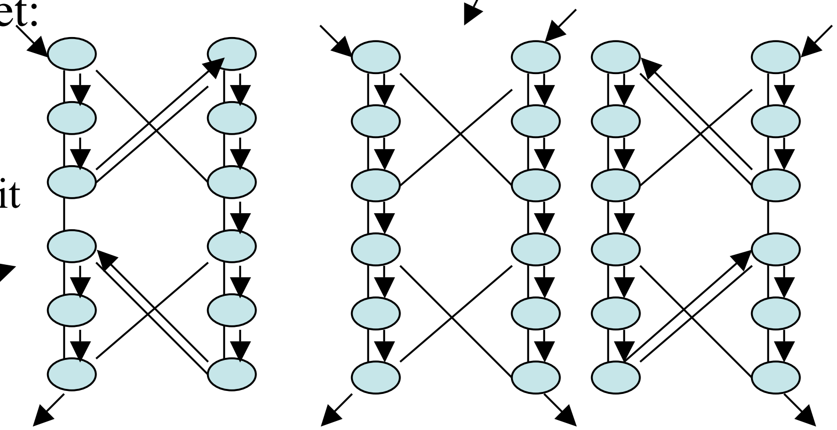
Proof cont’d

Middle one used if *u* and *v* are both in cover of G

We will make use of the following widget:



Here are some paths which visit each vertex in the widget



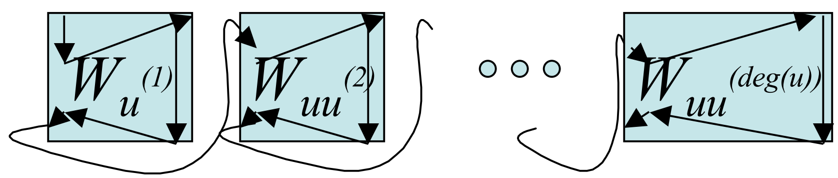
For each edge {*u, v*} in the original graph, the graph *G′* contains one copy of the widget W_{uv} (i.e, W_{uv} and W_{vu} are the same widget) and we denote the edges of the widget by [*u, v, i*] or [*v, u, i*] according to if they are on the left or right side. Only the tops and bottoms of widgets will be connected to the rest of the graph *G′*. In our construction, a cycle must visit each widget and there are exactly three different ways (as shown above) one could visit all the vertices of the widget: start on the left side, the right side, or do the two sides separately. In addition to the vertices of the widgets, we will have selector vertices, s_1,.., s_k. The edges chosen in these selector vertices will correspond to the *k* vertices of the vertex cover in *G*. We also have two additional types of edges besides those in the widgets that we describe on the next slide.

Yet More Proof.

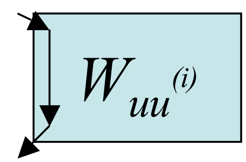
- For each $u \in V$ of G we add edges to form a path containing all widgets corresponding to edges incident on u in G. To do this we add the edges :

  $$\{([u, u^{(i)}, 6], [u, u^{(i+1)}, 1] \mid u \in V\}$$

  So we can construct a path from $[u, u^{(1)}, 1]$ to $[u, u^{(deg(u))}, 6]$ using these additional edges.



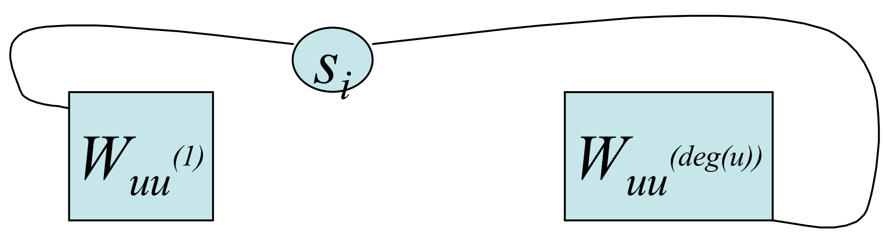
If both u and $u^{(i)}$ are in a vertex cover of G then traverse as



- The second kind of additional edges are of the form

  $$\{(s_i, [u, u^{(1)}, 1]) \mid \text{u is in v and } 1 \le j \le k\} \cup$$

  $$\{(s_i, [u, u^{(deg(u))}, 6] \mid u \text{ is in } v \text{ and } 1 \le j \le k\}.$$



Even More Proof.

- If $G=(V, E)$ then notice the size of a widget is constant and we have $|E|$ widgets.
- We also have only k selector vertices.
- There are at most sum of the degrees vertices of the first type
- There are at most $2k|V|$ additional edges of the second type.
- So in all the new graph G' will be polynomial size in G.
- Suppose G has a vertex cover $\{u_1,.. u_k\}$. A Hamiltonian cycle in G' can be obtained by starting at i=1 and for each i thereafter follow s_i to $[u_i, u_i^{(1)},1]$ and then the path from the previous slide to $[u_i, u_i^{(\deg(u\_i)},6]$. Then from here one can follow the edge $\{s_{i+1}, [u_i, u_i^{(\deg(u\_i)}, 6]\}$. Finally, one can following the edge $\{s_1, [u_k, u_k^{(\deg(u\_k)}, 6]\}$ back to the start.
- Since each edge in G is incident with one vertex in the vertex cover each widget will have all of vertices hit by this path if there is cover.
- On the other hand, if there is a hamiltonian cycle in G' then

$$V^* =\{ u \in V \mid \{s_j, [u, u^{(1)}, 1]\} \text{ is in the cycle for some } 1 \leq j \leq k\} \}$$

will be a vertex cover of size k in G.

The Traveling Salesman Problem

- In this problem a salesman must visit n cities. Between each pair of cities $\{i, j\}$ there is a cost c_{ij}.
- We want to know if it is possible for the salesman to see each city exactly once (except twice for the start city) with cost less than k?

$TSP = \{<G, c, k> \mid G$ is a complete graph c is the cost matrix, and k is an integer such that the travelling salesman has a tour of cost at most $k\}$

Theorem. *TSP* is **NP**-complete.

Proof. First given a tour we can verify if it satisfies the desired properties in polynomial time. So it is in **NP**. To see completeness we reduce *HAM-CYCLE* to it. Given an instance $G = (V, E)$ of hamiltonian cycle, we build an instance of TSP as follows. We first let G' be the complete graph on the same vertices. Then we set $c_{ij} = 0$ if $\{i, j\}$ is in E and $c_{ij} = 1$ otherwise. Then $< G', c, 0>$ is in *TSP* iff G was in *HAM-CYCLE*.