(1) Prove that the power set of a set S is not of the same cardinality as S.

2 Sets have the same cardinality if they have the same size. 2 Sets have the Same Size if there is a bijection between them.

~~Proof:~~

Proof: Let $f: S \to P(S)$ be a supposed bijection. Assuming S is countable, we have some function $s(k)$ to list out its elements $s(0), s(1), s(2), \ldots$ An element $\{s(1), s(5), \ldots\} \in P(S)$ can be viewed as a binary sequence $(0,0,1,0,0,1,\ldots)$ where we have a 1 if $s(i)$ is in $P(S)$ and a 0 otherwise. So f satisfies the diagonalization theorem. A complement of the diagonal for f will still be in $P(S)$ but not mapped to by f.

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2. $\langle M, w\rangle = \langle\langle Q, \Sigma, \Gamma, \delta, \{q_0\}, \{q_{accept}\}, \{\}\rangle, 0\rangle$

in A → $M = ($ $Q = \{q_{accept}, q_0\}$,
$\Sigma = \{0\}$,
$\Gamma = \{0, ␣\}$,
δ: $\delta(q_0, 0) = (q_{accept}, 0, L)$, $\delta(q_0, ␣) = (q_{accept}, ␣, L)$,
start states $\{q_0\}$
accept s. $\{q_{accept}\}$,
reject s. $\{\}$ $)$

$\langle M, w\rangle = \langle\langle Q, \Sigma, \Gamma, \delta, \{q_0\}, \{\}, \{q_{reject}\}\rangle, 0\rangle$

not in A → $M = ($ $Q = \{q_{reject}, q_0\}$,
$\Sigma = \{0\}$,
$\Gamma = \{0, ␣\}$
δ: $\delta(q_0, 0) = (q_{reject}, 0, L)$, $\delta(q_0, ␣) = (q_{accept}, ␣, L)$,
start s. $\{q_0\}$
accept s. $\{\}$,
reject s. $\{q_{reject}\}$ $)$

3) $\langle M, w\rangle \in HALT_{TM}$ iff $\langle M' \rangle \in$ ~~[illegible]~~

Machine that

$L = \{\langle M\rangle \mid M$ halts on all inputs beginning with $1\}$

The following machine F computes a reduction f

F = " On input $\langle M, w\rangle$:

1. Construct M';

   M' = " On input x:

   1. If x is not in the form 1w, halt → accept
   2. Otherwise, shift by 1 to right and run w on M
   3. If M accepts, accept

So if M accept w then M' will halt on all inputs & viceversa

2. Output $\langle M' \rangle$

Prove

Rice's Theorem 2 properties

⓪ Must notice if could do this problem could do $L' = \{\langle M\rangle \mid M$ accepts all strings beginning with $1\}$

① Strings in language + strings not in language

② ~~[illegible]~~ $L(M_1) = L(M_2)$ for any 2 TM's implies either both in L' or both not in L'

① This property is satisfied since if let M be the machine w/c on all inputs ~~[illegible]~~ accepts, then $\langle M\rangle \in L'$. On the other hand, if let M be the machine w/c immediately goes into an infinite loop ~~[illegible]~~ on all inputs, then $\langle M\rangle \notin L'$.

② If M_1 & M_2 are s.t. $L(M_1) = L(M_2)$, they either both accept all strings beginning with 1 or both don't.

∴ Both properties needed for Rice's Thm hold, so L' and hence L is undecidable.

4. Consider the language $L=\{<M> \mid M$ is a TM such that $L(M)$ is context free$\}$. Prove this language is undecidable without appealing to Rice's Theorem.

Proof. Let R be a TM that decides L above. Using R we can build the following machine S for A_{TM} and hence get a contradiction:

S= "On input $<M,w>$ where M is a TM and w is a string

1. Construct the following TM M_2.
    a. M_2 = " On input x:
        i. If x has the form $a^n b^n c^n$, *accept*.
        ii. If x does not have this form, run M on input w and *accept* if M accepts w."
2. Run R on input $<M_2>$.
3. If R accepts, *accept;* if R rejects, *reject*."

5. Want to show E_{PTIME} is undecidable.

It suffices to give a mapping reduction from A_{TM} to $\bar{E}_{PTIME}$, since if $\bar{E}_{PTIME}$ is decidable so is E_{PTIME}.
Given an instance $\langle M, \omega \rangle$ w/c might be in A_{TM}, we compute a pair $\langle M', 2|y|^2 \rangle$.
Here M' recognizes the language

$L = \{ y \mid y$ is an accepting computation history of M on $\omega \}$

So L is not empty, and hence $\langle M', p \rangle$ is in $\bar{E}_{PTIME}$ iff there is an accepting computation history of M on ω, which in term means M accepts ω. The polynomial $2|y|^2$ is so if we construct M' carefully it has enough time to ck L.
For instance, checking legit start and end configuration can be done in linear time. Verifying each intermediate C_{i+1} follows from C_i can be done in quadratic time in $|y|$. (Basically, we have to scan back and forth b/w the two configuration $|C_i| \leq |y|$ times. We do this twice for all configurations w/c gives the bound.

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6.) GIVE a decidable language H such THAT
$HALT_{TM} = \{ \langle M,x \rangle \mid \exists w, \langle w,M,x \rangle \in H \}$

$H := \{ \langle w,M,x \rangle \mid$ w IS THE CODE OF a SEQUENCE
OF CONFIGURATIONS,
Each CONFIGURATION YIELDING THE NEXT
ACCORDING TO THE TRANSITION TABLE OF TM M
ON INPUT X
ALSO THE LAST CONFIGURATION IS ACCEPTING OR REJECTING
"IT HALTS"

7.) SHOW THAT THE VARIANT OF PCP WHERE WE REQUIRE THAT EACH TILE BE PLAYED AT MOST ONCE IS DECIDABLE

$PCP = \{ \langle P \rangle \mid$ P IS AN INSTANCE OF PCP WITH
a MATCH by ENFORCING EACH TILE CAN
ONLY BE USED ONCE $\}$

S = "ON INPUT OF PCP STRING
1.) Loop TO TRY EACH possible
Combination of EACH TILE.
2.) IF A ACCEPTING CONFIGURATION IS
NON DETERMINISTICALLY FOUND ACCEPT.
3.) IF NO ACCEPTING CONFIGURAT IS
FOUND REJECT.

7

Each tile can only be used once. Therefore a collection of n tiles can be checked in ~~ka~~ linear time.

The number of possible combinations is $n!$. Imagine that checks all combinations one at a time, accepting if there is a match and rejecting if it reaches the last combination without a match.

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8. a) A_{TM} is T-recognizable, therefore may not be co-T-recognizable, or else it would be decidable.

b) $\overline{A_{TM}}$ is co-T-recognizable, but not T-recognizable, by the proof above.

c) EQ_{TM} is not T-recognizable, by $A_{TM} \leq_m \overline{EQ_{TM}}$:

F = "On input $\langle M, w \rangle$, use an inequality-recognizer to see if a universal rejector is not equal to a TM R with M and w built in such that whether or not M accepts w determines whether R is a universal acceptor or rejector."
F decides A_{TM}.

Neither is EQ_{TM} co-T-recognizable, by $A_{TM} \leq_m EQ_{TM}$:

G = "On input $\langle M, w \rangle$, create a TM that is either a universal acceptor or rejector based on whether M rejects w, and use an equality-recognizer to see if this machine is equivalent to a known universal rejector."
G decides A_{TM}.

9. $HALT_{TM}$ is undecidable via Recursion Theorem

Turing Machine R decides $HALT_{TM}$ for the purpose of obtaining a contradiction. Construct TM B.

B = " on input w :
⇒ an encoding of a TM B and string w.
1. Obtain via Recursion Theorem own description ⟨B⟩
2. Run TM R on input ⟨B, w⟩
3
a. If R accepts, then enter an infinite loop
b. If R rejects, then immediatly HALT.

10. Theorem: Incompressible strings of ~~len~~ every length exist.

Proof: The number of strings of length n is 2^n. Each description is a binary string, so the number of description of length ~~len~~ less than n is at most the sum of the number of strings of each length up to $n-1$, or

$$1+2+4+8+\dots+2^{n-1} = (2-1)(1+2+4+8+\dots+2^{n-1}) = 2^n - 1$$

which is less than the number of string of length n. So some incompressible string of length n must exist.