More sets. Sequences, tuples, functions, relations.

CS154

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Outline

- Venn Diagrams, basic set operations
- Sequences and Tuples
- Functions and Relations

More on sets

- Order of elements doesn't matter for sets:

  $\{1,5\} = \{5,1\}$
- Repetitions also don't matter:

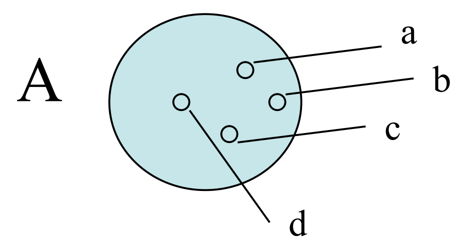
  $\{1,1,1,4,4,5\} = \{1, 4, 5\}$
- If we want repetitions to matter but still don't care about order then have a **multiset**.

Basic Set Operations

- Given two sets A and B, we can:
  - Take their **union**
    $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
  - Take their **intersection**
    $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
  - Take their **difference**
    $A/B = \{x \mid x \in A \text{ and } x \notin B\}$
- If we have a universe, U, under consideration, then taking the difference with respect to this set is called taking a **complement**. $\overline{A} = U \setminus A$.

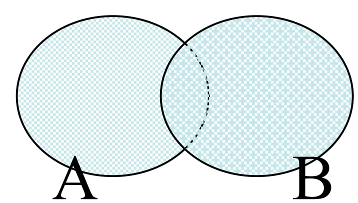
Venn Diagrams

- Sometimes it is useful to use Venn diagrams to represent or reason about sets.
- In such a diagram a set is represent by a circle or some other shape and points in the circle represent elements of this set



- By overlapping such sets and shading only the region one wants, one can represents sets operations. For example,

$A \cup B$



Tuples

- Given two objects, we write (a,b) for their ordered pair. This denotes the set

  {a, {a,b}}.

  Notice this is different then the set (b,a) which is

  {b, {b,a}}= {b, {a,b}}. So order matters.
- We can iterate this operation (a,b,c) =(a, (b,c)). This would be called an ordered 3-tuple.
- In general, one can have ordered k-tuples.
- Given two sets A, B their Cartesian Product AxB is: $\{(a,b) \mid a \in A \text{ and } b \in B\}$.
- We write A^k for $\overbrace{A \times A \times \ldots \times A}^{k \text{ times}}$.

Natural Numbers (formally)

- The natural numbers can be defined using only sets.
- We use the empty set $\varnothing$ for 0.
- Then given a set A, define S(A), the successor of A, to be the set
  $A \cup \{A\}$.
- Define 1 as $S(0) = \{\varnothing\}$, 2 as $S(1) = \{\varnothing, \{\varnothing\}\}$, …
- The natural numbers can be defined as the smallest set containing $\varnothing$ and closed under successor.
- Using pairing could now define $\mathbb{Z}$ using just sets

Sequences

- A sequence is an ordered list of objects.
  - So an ordered pair is a sequence
  - Any k-tuple for any k will be a sequence.
  - We will also allow infinite sequences.

Power Set

- Given a set A, we define its power set, P(A), to be the set of all subsets of A.
- For example, if A={a,b,c}, then P(A) is

  $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

Functions

- A function associates the values of some input set called the **domain**, with some output set called the **range**.
- We write f: A->B to say f is a function or mapping from A to B.
- We write f(a)=b to say that f maps the element a of A to b of B.
- We can view a function as a set of pairs of the form (a,b), where we have one and only pair for each element of A.

More Functions

- Succ: $\mathbb{N} \longrightarrow \mathbb{N}$

  Succ(x) = x+1

  Notice each value of the range there is at most one element that maps to it. Such a function is called **injective** or **one-to-one**.
- Add: $\mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$

  Add(x,y) = x+y

  Notice for each element x in the range there is some element which maps to it. (For instance, (x,0)). Such a function is called **surjective** or **onto**.
- A function which is both one-to-one and onto is called a **bijection**.
- Notice Add takes two arguments. The number of arguments to a function is called its **arity**. We would call Add a binary function. In general, can have **k-ary** functions.
- Note some functions such as +, *, - are typically written in **infix** notation(i.e., x-y) rather than **prefix** notation -(x,y).

Relations

- A **predicate** or **property** is a function whose range is {TRUE, FALSE}.
- A predicate whose domain is a set of k-tuples is called a **relation**.
- As with functions, one can have binary, k-ary relations.
- As an example of a relation one might have

  Less: $\mathbb{N} \times \mathbb{N} \longrightarrow \{\text{TRUE, FALSE}\}$

  Less(x,y) = TRUE if x<y and FALSE otherwise.

Equivalence Relations

- One particularly useful kind of relation is an **equivalence relation**. Such a relation acts like ‘=’.
- A binary relation R is an equivalence relation if for each x,y,z:
  - R is **reflexive**, that is, xRx. (xRx is just R written in infix and we write xRx to mean xRx = TRUE).
  - R is **symmetric**, that is, xRy implies yRx
  - R is **transitive**, that is, xRy and yRz implies xRz.
- For example congruent mod n is an equivalence relation.