Decidable Languages and Diagonalization

CS154

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Apr 5, 2006.

Outline

- More decidable problems for CFLs
- Universal Turing Machines
- Diagonalization

CFL Emptiness and Equality

- Let E_{CFG}={<G> | G is a CFG such that L(G) is empty}.

Theorem E_{CFG} is decidable.

Proof. Let S be the following Turing machine:

T= “On input <G>, where G is a CFG:

1. Mark all terminal symbols in G.
2. Repeat until no new variables get marked:
   a) Mark any variable A where G has a rule A-->U_1,..,U_k provided all the U_i are already marked.
3. If the start variable is not marked, accept; otherwise, reject.”

- Let EQ_{CFG}={<A,B> | A, B are CFGs and L(A)=L(B)}
- It turns out EQ_{CFG} is not decidable, but we will show this a fair bit later.

CFL Decidability

Theorem Every CFL is decidable.

Proof. Let L be a CFL and let G be a grammar that generates it. First, we can use the algorithm from the March 1 lecture, to convert this to Chomsky Normal Form. Call the resulting grammar G'. Then we construct a TM $M_{G'}$ which operates as follows:

$M_{G'}$="On input w:

- Simulate the CYK algorithm on w according to G'.
- If the algorithm accepts, then accept; otherwise, reject."

Universal Turing Machines

- Continuing in the vein of the last couple of lectures, it is natural to ask if there is a decision procedure for:

A_{TM}={ <M,w> | M is a TM and M accepts w}

- There is a recognition procedure for this language:

U=“ On input <M,w>, where M is a TM and w is a string:

1. Simulate M on input w.
2. If M ever enters its accept state, accept; if M ever enters its reject state, reject.”

- The above Turing Machine is called a **Universal Turing Machine (a UTM)** because it can be used to simulate any other Turing machine.
- However, as U on a given input does not necessarily halt, it is not a decision procedure for A_{TM}.
- It turns out it is impossible to get a decision procedure for A_{TM}.
- The next few slides work towards showing this.

Sizes of Sets

- In the 1870's Georg Cantor was interested in figuring out when two sets are of the same size.
- In particular, he was worried about infinite sized sets.
- He argued two sets A, B should be said to be of the same size if there is a one-to-one, onto function (a **bijection**) between them.
- Recall **one-to-one** means a ≠ b implies f(a) ≠ f(b) and **onto** means for every element b in B, there is some a in A such that f(a) = b.
- For example the map f(k)=2k is a bijection between the integers and the even integers.
- A set is said to be **countable** if there is a bijection between it and a subset of the naturals. Otherwise, a set is said to be uncountable.
- For example, the rational numbers and the set of finite strings over are {0,1} are countable. (will doodle on board why, but also see book).

The Diagonal of a Function on Sequences.

- Suppose f is a one-to-one function from a countable set A={a(0), a(1), a(2), …} to sequences of elements over some set B of size at least 2.
- For example,

  f(a(0)) = (1, 0, 1)

  f(a(1)) = (0, 0, 0)

  f(a(2)) = (0, 1, 1)
- Let $f(a(i))_j$ denote the jth element of the sequence f(a(i)).
- The diagonal of this function is the function of f is the sequence $d(f)=(f(a(0))_0, f(a(1))_1, f(a(2))_2,\ldots)$.
- So in this case d(f) = (1, 0, 1).
- Call a sequence d'(f) a **complement** of the diagonal if $d'(f)_i$ is always different from $d(f)_i$.
- For example, for the f above a possible d'(f) is (0, 1, 0).
- The following theorem is an easy consequence of our definition.

Theorem (Diagonalization Theorem) If f satisfies the first bullet above then it does not map any element to a complement of its diagonal.

Corollaries to the Diagonalization Theorem

Corollary. Countable set A is not the same size as its P(A).

Proof. Let f:A --> P(A) be a supposed bijection. Since A is countable, we have some function a(k) to list out its elements a(0), a(1), a(2), …An element {a(2), a(5), ..}∈P(A) can be view as a binary sequence (0, 0, 1, 0, 0, 1, …) where we have a 1 if a(i) is in P(A) and a 0 otherwise. So f satisfies the diagonalization theorem. A complement of the diagonal for f will still be in P(A) but not mapped to by f.

Corollary. The reals are uncountable.

Proof. Consider the function g:R-->(0,1) given by

$$g(x) = 1/2[x/(1+|x|) +1].$$

It is not hard to see this is one-to-one and onto. So it suffices to show the open interval (0,1) is uncountable. Let f:N --> (0,1) be a supposed bijection between (0,1) and this interval. A number x ∈ (0,1) be viewed as a decimal point followed by sequence over 0-9. Pick a complement of the diagonal that has no 0's or 9's. This will again be a number in (0, 1) but not mapped to by f.