B-spline and surfaces

CS116B
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Outline

- B-Spline Curve Equations
- Cubic Periodic B-Splines
- Open Uniform B-Splines
- Nonuniform B-Splines
- B-Spline Surfaces

B-Splines

- Commonly implemented in many graphics packages.
- Have two main advantages over Bezier splines:
  - the degree of a B-spline can be set independently of the number of control points.
  - B-splines allow better local control over the shape of the spline.
- However, are slightly more complicated than Bezier curves.

B-Spline Curve Equations

- Assume have n+1 control points $\mathbf{p}_k=(x_k,y_k,z_k)$.
- The B-Spline curve of degree $1<d<=n+1$ is given by:

$$\mathbf{P}(u) = \sum^{n}_{k=0} \mathbf{p}_k B_{k,d}(u) \text{ for } u \text{ in } [u_{min},u_{max}]$$

- $B_{k,d}$ will actually be of degree d-1. It is given by the equations:

$$B_{k,1}(u) = 1 \text{ if } u \text{ in } [u_k,u_{k+1}]$$
$$= 0 \text{ otherwise}$$

$$B_{k,d}(u) = (u-u_k)/(u_{k+d-1}-u_k)B_{k,d-1}(u)$$
$$+(u_{k+d}-u)/(u_{k+d}-u_{k+1})B_{k+1,d-1}(u)$$

More B-Spline Equations

- Each subinterval endpoint u_j is called a **knot**.
- The set of subinterval endpoints is called a **knot vector**.
- Any terms of the form 0/0 are assumed to evaluate to 0.

B-spline Properties

- Curve has degree d-1 and C^{d-2} continuity
- For n+1 control points curve described with n+1 blending functions
- Each blending function $B_{k,d}$ is defined over d subintervals of the total range of u, starting at the knot value u_k.
- The range of the parameter u is divided into n+d subintervals by the n+d+1 values specified in the knot vector.
- Given a vector $\{u_0, u_1, \ldots u_{n+d}\}$, the spline is only specified between u_{d-1} and u_{n+1}.
- Each section of the spline curve is influenced by at most d control points
- Any control point affects at most d curve sections.
- $\sum^{n}_{k=0} B_{k,d}(u)=1$, so have a convexity property like with Bezier curves.

Types of B-Spline

- Uniform --equal spacing between knots
- Open-Uniform -equal spacing except at the end
- Nonuniform

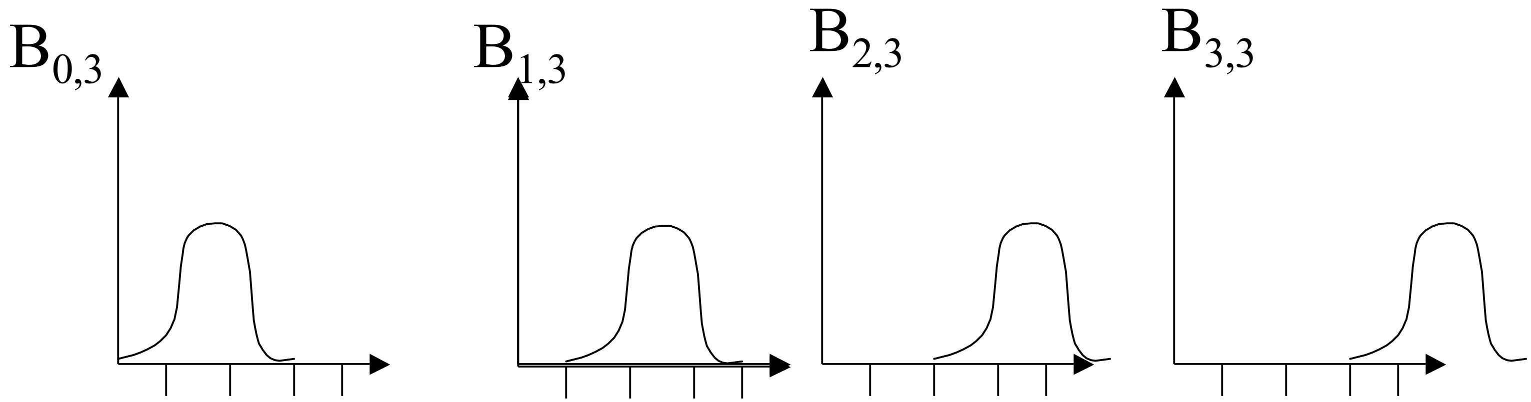
Uniform B-Splines

- For these the spacing between knots is constant. For example, might have:

  {-1.5, -1., -.5, 0.0, .5, 1.0, 1.5, 2.0}
- Often choose values between 0 and 1, {0, .25, .5, .75, 1}.
- Or choose spacing which are integers {0, 1, 2, 3…}
- Uniform B-splines have periodic blending functions. So:

  $B_{k,d}(u) = B_{k+1,d}(u+\Delta u) = B_{k+1,d}(u+2\Delta u)$

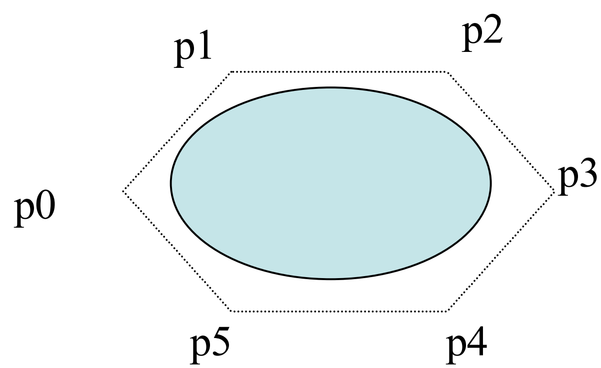
Example



- Suppose n=d=3 and the n+d+1=7 knot values are {0, 1, 2, 3, 4, 5, 6}.
- Book works out $B_{k,3}$ for each k.

Cubic Periodic B-splines

- Theses are the most common periodic B-Splines used in software packages
- They are useful for making closed curves. For example, consider:



- Could cyclically specify four of 6 points as control points to get a cubic spline for a section of figure above

More Cubic Curve Equations

- To derive the curve equations for control points $\mathbf{p_0}$, $\mathbf{p_1}$, $\mathbf{p_2}$, $\mathbf{p_3}$ the cubic B-spline can start from the boundary conditions:

  $\mathbf{P}(0)= 1/6(\mathbf{p_0} + 4*\mathbf{p_1} + \mathbf{p_2})$

  $\mathbf{P}(1) = 1/6(\mathbf{p_1} + 4*\mathbf{p_2} +\mathbf{p_3})$

  $\mathbf{P}'(0) = 1/2(\mathbf{p_2} - \mathbf{p_0})$

  $\mathbf{P}'(1) = 1/2(\mathbf{p_3} - \mathbf{p_1})$

- Notice these are similar to the cardinal spline conditions.
- Similarly, have a matrix representation:

  $\mathbf{P}(u) = [u^3\ u^2\ u\ 1]\mathbf{M_B}[\mathbf{p_0}\ \mathbf{p_1}\ \mathbf{p_2}\ \mathbf{p_3}]^t$ where $\mathbf{M_B}$:=

$$\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

Open Uniform B-Splines

- Spacing of knots is uniform except at the ends where the same value is repeated d times.
- For example, the knot vector {0,0, 1, 2, 3, 3} is a possibility when d=2 and n=3
- These splines have characteristics similar to Bezier splines.
- In fact, where d=n+1 and all the control values are 0 or 1 then get a Bezier curve.
- For example when d=4, knot vector would be {0,0,0,0,1,1,1,1}

Nonuniform B-Spline Curves and B-Spline Surfaces

- If the control points are not evenly spaced then have a non-uniform B-spline curve.

- If have a grid of values can also adapt B-spline, like we did for Bezier curves, to get B-Spline surfaces.