On (1,2)-Strongly Indexable Spiders

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ABSTRACT

For any integers $k, d \geq 1$, a (p,q)-graph G with vertex set $V(G)$ and edge set $E(G)$, $p=|V(G)|$ and $q=|E(G)|$, is said to be (k,d)-strongly indexable (in short **(k,d)-SI**) if there exists a function pair (f, f^+) which assigns integer labels to the vertices and edges, i.e., $f: V(G) \rightarrow \{0,1,\ldots,p-1\}$ and $f^+: E(G) \rightarrow$□$\{k,k+d, k+2d,\ldots,k+(q-1)d\}$ are onto, where $f^+(u, v) = f(u)+f(v)$ for any $(u, v) \in E(G)$. We determine here classes of spiders that are (1,2)-SI graphs. We show that every given (1,2)-SI spider can extend to an (1,2)-SI spider with arbitrarily many legs.

1. Introduction. In 1990, Acharya and Hegde [2] have introduced the concept of strongly k-indexable graphs: A (p, q)-graph $G = (V;E)$ with p vertices and q edges is said to be **strongly k-indexable** if its vertices can be assigned distinct numbers $0,1, 2, \ldots,p-1$ so that the values of the edges, obtained as the sums of the numbers assigned to their end vertices form an arithmetic progression $k, k + 1, k + 2,\ldots, k + (q - 1)$. When $k = 1$ strongly k-indexable graph is simply called strongly indexable graph. Later, they extend the concept to the following

Definition 1.1. For any integers $k, d \geq 1$, a graph G with vertex set $V(G)$ and edge set $E(G)$, $p=|V(G)|$ and $q=|E(G)|$, is said to be **(k, d)- strongly indexable** (in short **(k, d)-SI**) if there exists a function pair (f, f^+) which assigns integer labels to the vertices and edges, i.e.,

$f: V(G) \rightarrow \{0,1,\ldots,p-1\}$ and $f^+: E(G) \rightarrow \{k,k+d,k+2d,\ldots,k+(q-1)d\}$ are onto, where $f^+(u, v) = f(u)+f(v)$ for any $(u, v) \in E(G)$.

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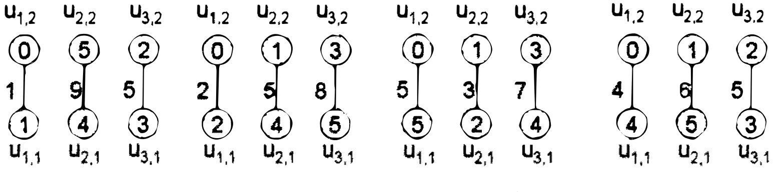
Thus strongly k-indexable graph are (k, 1)-strongly indexable and strongly indexable graph is (1, 1)-strongly indexable.

If we relaxed the definition of f in strongly (k, d)-indexable graph by $f: V(G) \rightarrow N$, then we have the concept of (k, d)-arithmetic graphs of Acharya and Hegde [1].

For any $k,d \geq 1$, we denote the class of all (k,d)-SI graphs by **$\Omega(k,d)$**.

Example 1. Figure 1 shows that the disconnected graph $3K_2$ is (1,4)-, (2,3)-, (3,2)-, and (4,1)-SI.



(1,4)-SI (2,3)-SI (3,2)-SI (4,1)-SI

Figure 1. The forest $3K_2$ admits different (k,d)-SI labelings.

Example 2. The following are two different (1,1)-SI labelings of $K_2 x C_3$.

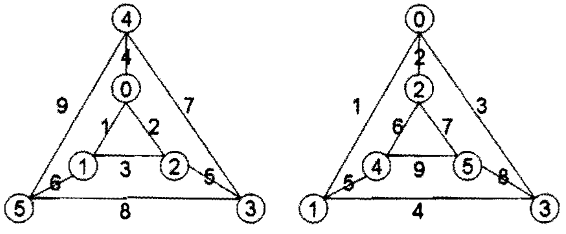


Figure 2. $K_2 x C_3$ has different (1,1)-SI labelings.

Example 3. The tree $CT(3;3^{[3]})$ is (7,1)-SI and (10,1)-SI.

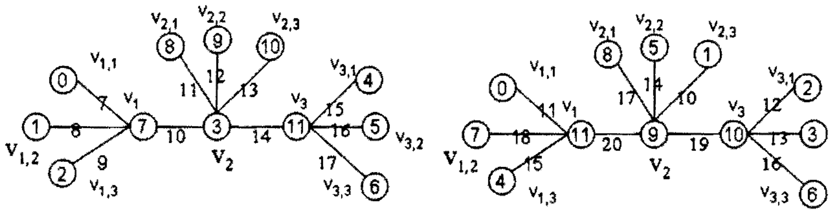


Figure 3. Tree which is (7,1)-SI and (10,1)-SI.

Acharya and Hegde showed that the only non-trivial regular graphs that are strongly indexable are K_2, K_3 and $K_2 \times K_3$.and that every strongly indexable graph has exactly one non-trivial component that is either a star or a triangle. Results on strongly indexable graphs are meager. There are few examples of strongly indexable graphs were known. There are many interesting questions left open.

In [7], it is shown that

Theorem 1.1. The caterpillar T is (1,2)-SI if and only if its bipartition (M,N) has the property that $||M|-|N|| \leq 1$.

A tree is called a ***spider*** if it has a center vertex c with degree $x > 1$ while each of the other vertices is either a leaf or has degree 2. Thus, a spider is an amalgamation of k paths with various lengths. If it has x_1 paths with length a_1, x_2 paths with length a_2, etc., we denote the spider by $SP(a_1^{x1}, a_2^{x2}, \dots, a_m^{xm})$, where $x_1 + x_2 + \dots + x_m = x$. (See Figure 4.)

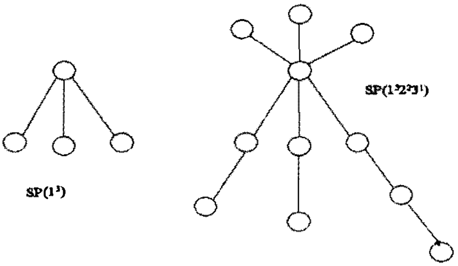


Figure 4.

General (k, d)-SI graphs were considered by the first author in [6]. Lee et al [7] determine classes of graphs that are (1, 2)-SI and (2, 2)-SI. We determine here classes of spiders that are (1,2)-SI.

2. (1,2) - SI Spiders with three legs.

Lemma 2.1. The path P_n has a natural (1,2)-SI labeling.

If $V(P_n) = \{v_1, v_2, \dots, v_n\}$, then the labeling $f(v_i) = i-1$ is clearly (1,2)-SI labeling.

Lemma 2.2. If n is even, then the path P_n has another (1,2)-SI labeling which is defined as follows:

$g(v_i) = i$ if i is odd,

and $g(v_i) = i-2$ if i is even.

We will call this labeling as twist(1,2)-SI labeling.

Example 4. Figure 5 shows P8 with natural and twist (1,2)-SI labelings.

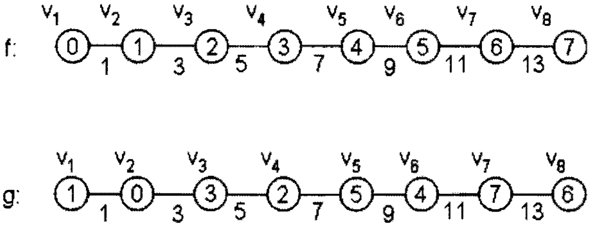
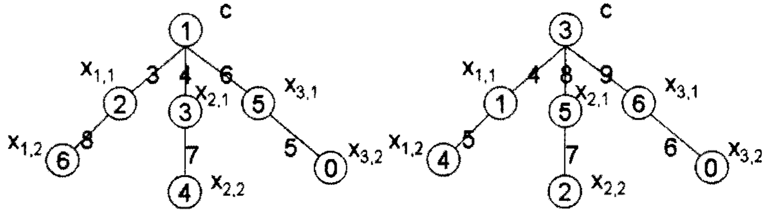


Figure 5.

The condition of Theorem 1.1. is not sufficient for spiders to be (1,2)-SI.

Example 5. Consider the spider SP(2,2,2) which is the spider with three legs of length 2. We see that it is (3,1)-SI and (4,1)-SI (see Figure 5). However, it is not (1,2)-SI.



SP(2,2,2) is (3,1)-SI SP(2,2,2) is (4,1)-SI

Figure 6. Spider SP(2,2,2) is (3,1)-SI and (4,1)-SI.

The following result provide an infinite many (1,2)-SI spiders with three legs.

Theorem 2.2: For $n \geq 2$, and $m \geq n$ the spider SP(n, m,m+1) is (1,2)- SI .

Proof. SP(n, m, m+1) has n+m+m+1+1 = n+2m+2 vertices and n+2m+1 edges. We need to prove that there is vertex labeling

$f: V(SP(n,m,m+1)) \rightarrow \{0,1,2,\ldots,n+2m+1\}$ with the induced edge labeling $f^{+}(E(SP(n,m,m+1))) \rightarrow \{1,3,5,\ldots,2(n+2m+1)-1\}$.

Let us denote the vertices of SP(n, m, m+1) as in figure below (Figure 7) :

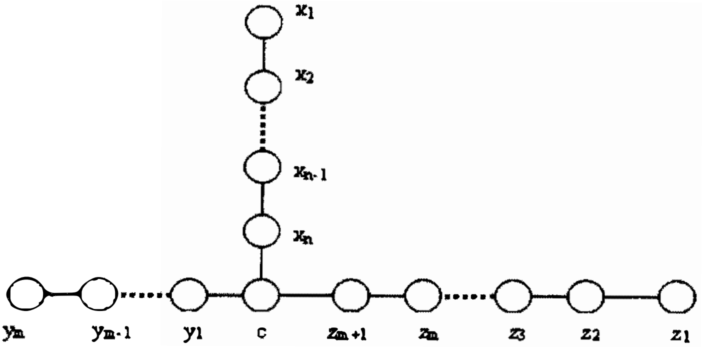


Figure 7.

We will label the vertices $x_1, x_2, \ldots, x_n$ with 0, 1, ..., n-1. center vertex c with n, label $y_1, y_2, \ldots, y_m$ with n+1, n+2, ..., n+m, and label $z_1, z_2, \ldots, z_{m+1}$ with n+m+1, n+m+2, ..., n+m+m+1, respectively. i.e. $f{:}V(G) \rightarrow \mathbf{Z}_{n+2m+2}$ is

$f(x_i) = i-1$ for $i=1,2,..,n$.
$f(y_i) = n+i$ for $i=1, 2, \ldots, m$.
$f(c)=n$,
$f(z_i)= n+m+i$ for $i=1, 2,\ldots, m+1$.

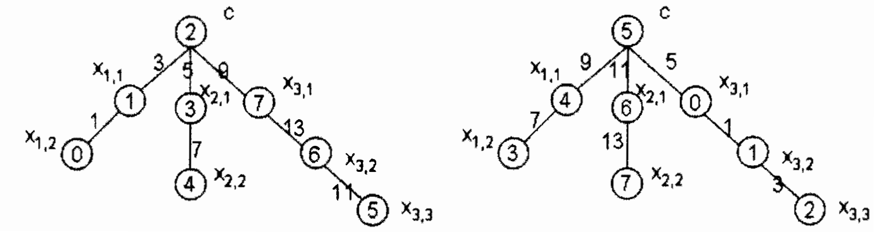
Now let us check the induced edge labels. It can be seen that

$f^+(\{x_i, x_{i+1}\}) = 2i-1$, for $i=1,2,\ldots,n-1$.
$f^+(\{x_n, c\})=2n-1$,
$f^+(\{c, y_1\})=2n+1$,
$f^+(\{y_i, y_{i+1}\}) = (n+i)+(n+i+1)=2n+2i+1$, for $i=1, 2,\ldots,m-1$.,
$f^+(\{c, z_{m+1}\}) = n+(n+m+m+1)=2n+2m+1$,
$f^+(\{z_i, z_{i+1}\}) = (n+m+i)+(n+m+i+1)=2n+2m+2i+1$, for $i=1,2,\ldots,m$.

From the above, we can see f^+ has range

$$R= \{2i-1: i=1,2,\ldots,n-1\} \cup \{2n-1, 2n+1\} \cup \{2n+2i+1: i=1,2,\ldots,m-1\} \cup \{2n+2m+1\} \cup \{2n+2m+2i+1 : i=1,2,\ldots,m\}$$
$$= \{1,3,5,\ldots,2n-3,2n-1,2n+1,2n+3,\ldots,2n+2m-1, 2n+2m+1, 2n+2m+3, \ldots ,2n+4m+1\}.$$□

Example 6. Spider SP(2,2,3) with two different (1,2)-SI labelings.



SP(2,2,3) is (1,2)-SI

SP(2,2,3) is (1,2)-SI

Figure 8.

We have shown in [6] a general construction of (k, d)-SI graph from two given (k, d)-SI graphs. We illustrate here the usefulness of this method by presenting a recursive construction of infinite families of (1, 2)-SI spiders with three legs.

Ingredient: Suppose G is a (p_1,q_1)-graph in $\Omega(k_1,d)$ and H is a (p_2,q_2)-graph in $\Omega(k_2,d)$ with labelings g,h respectively.

Constraint: d is a divisor of $2p_1+(k_2-k_1)$ and $[2p_1+(k_2-k_1)]/d - q_1 \geq 0$.

We can construct a new graph on $V(G) \cup V(H)$ as follows:

Keep the original (k_1,d)-labeling on G and extend the vertex labeling on H by $h\oplus p_1$ where $(h\oplus p_1)(v) = h(v)+p_1$ for all $v\in V(H)$.

Under the $h\oplus p_1$ labeling H becomes a $(2p_1+k,d)$-SI graphs.

Let $t = [2p_1+(k_2-k_1)]/d - q_1 \geq 0$.

If t =0, then the disjoint union $G \cup H$ is (k_1, d)-SI.

If t >0, let us fill in t edges which connect vertices of G and H by the following scheme :

Pick u in G with label x and v in H with label $2p_1+y$ join them so that its induced edge label $2p_1+x+y$ is range from k_1+q_1d to $k_1+(q_1+1)d, \ldots, k_1+(q_1+t-1)d$. We denote the set of these edges by Π. That is $\Pi = \{(u,v): g(u)=x$ and $h(v)=y$ and $x+y= k_1+q_1d, k_1+(q_1+1)d, \ldots, k_1+(q_1+t-1)d\}$.

Then $E(G) \cup E(H) \cup \Pi$ is (k_1,d)-SI.

We denote this graph by $G\oplus\Pi\oplus H$.

Theorem 2.2. If G is a (p_1,q_1)-graph in $\Omega(k_1,d)$ and H is a (p_2,q_2)-graph in $\Omega(k_2,d)$ and d is a factor of $2p_1+(k_2-k_1)$ with $[2p_1+(k_2-k_1)]/d - q_1 \geq 0$, then there exists a $(p_1+p_2, q_2+[2p_1+(k_2-k_1)]/d)$ graph in $\Omega(k_1,d)$ which contains G, H as induced subgraphs.

Theorem 2.3. For any (p_1, q_1)-graph G, (p_2,q_2)-graph H in $\Omega(k, 2)$, with $p_1\geq q_1$ we can construct a (k, 2)-SI graph which contains G,H as induced subgraph.

Now let us consider k=1. We will illustrate the above construction by the following example.

Example 5. Using $G = P_7$, $H= P_5$ and $\Pi = \{(x_{1,3}, y_{1,5})\}$. We see that $G\oplus\Pi\oplus H$ =SP(2,4,5) is (1,2)-SI.

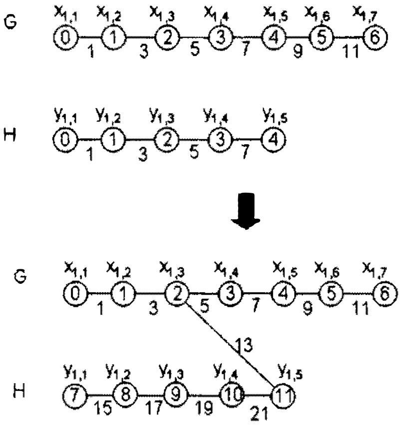


Figure 9.

Theorem 2.3: For $n \geq 1$, and $m \geq 1$, the spider SP(n, n+1,m) is (1,2)- SI .
Proof. Let $G = P_{n+m+1}$ and $H = P_{m+1}$ with the natural (1,2)-SI labeling **and** $\Pi = \{(x_{1,1}, y_{1,m})\}$. We see that $G \oplus \Pi \oplus H$ =SP(n,n+1,m) is (1,2)-SI.□

Corollary 2.4: For $k \geq 1$, the spider SP(1,2,k) is (1, 2)- SI .

Proof: Let us label the vertices of SP(1,2,k) as in the figure below:

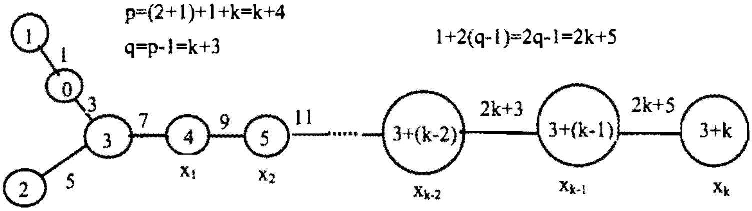


Figure 10.

It is clear from the figure above that SP(1,2,k) has p=k+4 vertices and q=k+3 edges and the vertex labeling induces the edge labeling of {1, 3, 5, …, 2(q-1)} = {1, 3, 5, …, 2k+5}. This proves SP(1,2,k) is (1,2)-SI for any positive integer.

Theorem 2.5: For $k \geq 1$, the spider SP(2k,2k,2k+2) is (1, 2)- SI .

Proof. Let $G = P_{4k+1}$ with the natural (1,2)-SI labeling and $H = P_{2k+3}$ with the reverse twist (1,2)-SI labeling **and** $\Pi = \{(x_{1,2k+1}, y_{1,1})\}$. We see that $G \oplus \Pi \oplus H$ =SP(2k,2k,2k+2) is (1,2)-SI.□

Example 6. Spider SP(4,4,6) with its (1,2)-SI labelings.

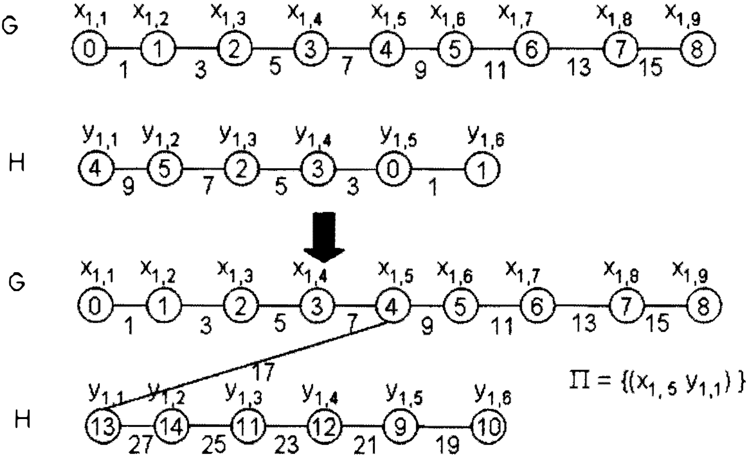


Figure 11.

3. (1,2) - SI Spiders with more than three legs.

Theorem 3.1. The spider SP($1^{[n]}$,2,2) is (1,2)-SI if and only if n=1 and 2.
Proof. If n=1 and n=2, we see that SP(1,2,2) and SP(1,2,2,2) are (1,2)-SI.

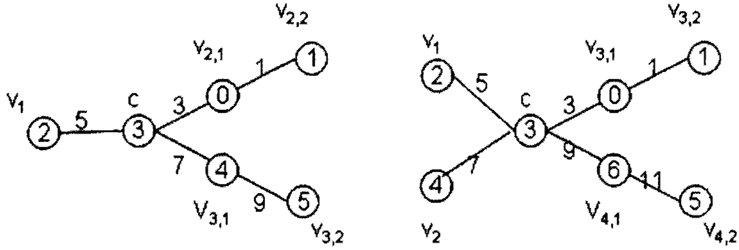
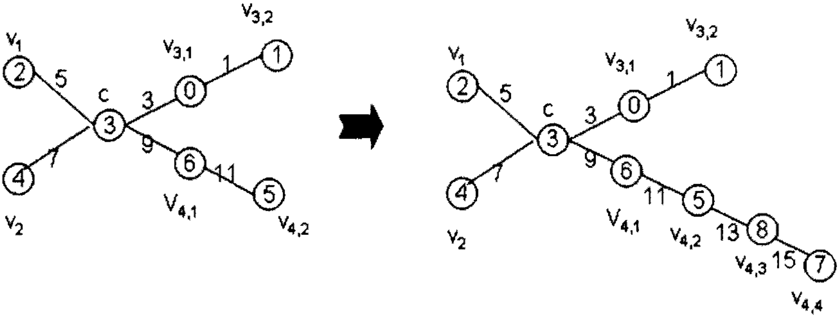


Figure 12.

However, if n≥3, then the bipartition (M,N) of the spider SP($1^{[n]}$,2,2) has the property that $||M|-|N|| > 1$. Therefore SP($1^{[n]}$,2,2) is not (1,2)-SI.□

Theorem 3.2. The spider SP(1,1,2,2k) is (1,2)-SI for all k≥1.
Proof. For k=1, we see in Theorem 3.1. that it is (1,2)-SI.
Assume the statement is true for k =n, i.e. SP(1,1,2,2n) is (1,2)-SI. We want to show that SP(1,1,2,2n+2) is also (1,2)-SI. We can extend SP(1,1,2,2n) to SP(1,1,2,2n+2) by adding two vertices $\{x_{4,2k+1}, x_{4,2k+2}\}$ and two edges $(x_{4,2k}, x_{4,2k+1})$, $(x_{4,2k+1}, x_{4,2k+2})$. Now we extend the original (1,2)-SI labeling f of SP(1,1,2,2n) to SP(1,1,2,2n+2) by setting

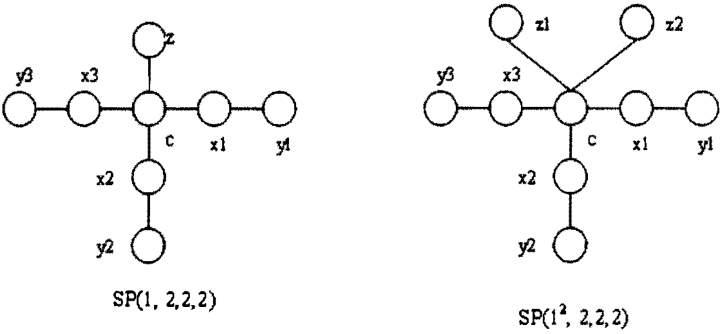
$$f(x_{4,2k+1}) = 2k+4 \quad , \qquad f(x_{4,2k+2}) = 2k+3.$$



. **Figure 13.**

It is clear that this is a (1,2)-SI labeling (see Figure 13.■

Theorem 3.3 The spider $SP(1^{[n]},2^{[3]})$ is not (1,2)-SI for all n.
Proof: First we show that for n=1 and 2, $SP(1^{[n]},2^{[3]})$ is not (1,2)-SI. .For easier describe the labelings of vertices, let us denote the spiders as



SP(1, 2,2,2)

$SP(1^2, 2,2,2)$

Figure 14.

For SP(1,2,2,2): c can have odd or even label.

(I) c has even label. Then x1, x2, x3 and z must have odd labels.

1) c has label 0.

(i) z has label 1. So {3, 5, 7} are labels of x1, x2 and x3 and it makes no difference which one has which, so say x1 has label 3, x2 has label 5 and x3 has label 7. So y3 must have label 6. Now no vertex can have label 2. Since if y1 were 2, then edges (x1, y1) and (c, x2) will have label 5; if y2 were 2, then edges (x2, y2) and (c, x3) both have label 7. Hence this is not a Q(1,2)-VG labeling.

(ii) z has label 3. Similar as above. No vertex can have label 2.

(iii) z has label 5. Similar as above. No vertex can have label 2.

(iv) z can not have label 7. since no way to get edge label 13.

2) c has label 2.
   (i) z can not label 1. Since no way to get edge label of 1.
   (ii) z has label 3. So $\{1, 5, 7\}$ are labels of $\{x_1, x_2, x_3\}$. Since vertex of label 0 has to be adjacent to vertex of label 1 to generate edge label of 1 and vertex of label 6 has to be adjacent to vertex of label 7 to generate edge label 13. This means vertex of label 4
   (iii) z has label 5. Similar as above. No vertex can have label 4.
   (iv) z can not have label 7. since no way to get edge label 13.

3) c has label 4.
   (i) z can not be 1, since no edge will have 1.
   (ii) z has label 3, 5 or 7, then x1 has label 1 and y1 must have label 0. Then no way to get edge label 3.

4) c has label 6. Similar as case 3). x1 must be 1 and y1 must be 0. Hence no way to generate edge label 3 again.

(II) c has odd label. Then x_1, x_2, x_3 and z must have even labels.
   (i) c has label 1. z cannot not be 0. Since 7 has to be adjacent to 6 to generate edge label 13. If z is 0, then no vertex can have 3. If z is nonzero, then x1 is 0, then y1 cannot have any label.
   (ii) c has label other than 1, then x1 must be 0 and x2must be 6 and y1 must be 1 and y2 must be 7 to generate edge label 1 and 13, respectively. This means y3 has to be 5. Then x3 cannot have a label 2 or 4.

If $n \geq 2$, then the bipartition (M,N) of the spider $SP(1^{[n]},2^{[3]})$ has the property that $||M|-|N|| > 1$. Therefore $SP(1^{[n]},2,2)$ is not (1,2)-SI.□

Example 11. Spider SP(1,2,2,2) is (3,1)-SI, (4,1)-SI and (5,1)-SI but not (1,2)-SI.

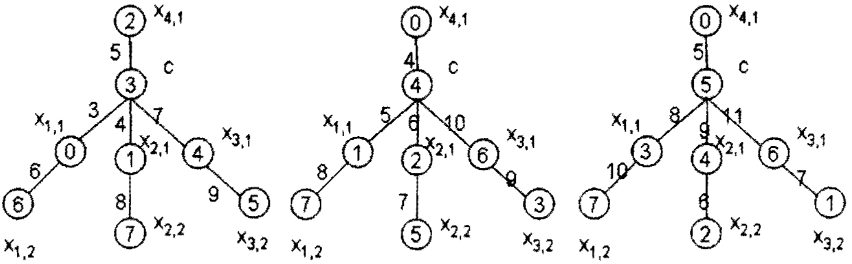
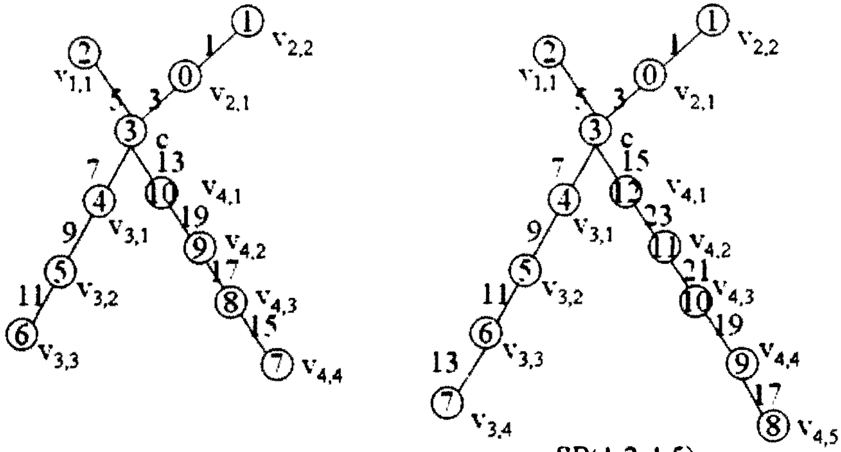


Figure 15.

However, we see

Theorem 3.3. The spider SP(1,2,n,n+1) is (1,2)-SI for all $n \geq 1$.
Proof. Let G = SP(1,2,n) with the (1,2)-SI labeling as Corollary 2.4.and $H = P_{n+1}$ with the reverse twist (1,2)-SI labeling **and** $\Pi = \{(x_{1,2k+1}, y_{1,1})\}$. We see that $G \oplus \Pi \oplus H$ =SP(2k,2k,2k+2) is (1,2)-SI.□

Example 12. Figure 16 illustrates the labeling scheme for n=3 and 4.



SP(1,2,3,4) SP(1,2,4,5)

Figure 16.

Theorem 3.4. The spider SP($1^{[n]}$,$2^{[2]}$,3) is (1,2)-SI if and only if n=1.
Proof. If n =1, Figure 17 depicts a (1,2)-SI labeling for spider SP(1, $2^{[2]}$,3).

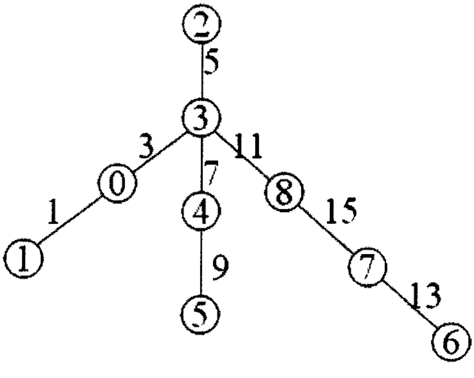


Figure 17.

We want to show that the spider SP($1^{[n]}$,$2^{[2]}$,3) is not (1,2)-SI for n>1.For the bipartition (M,N) of SP($1^{[n]}$,$2^{[2]}$,3) is $||M|-|N|| > 1$. Therefore SP($1^{[n]}$,$2^{[2]}$,3) is not (1,2)-SI.□

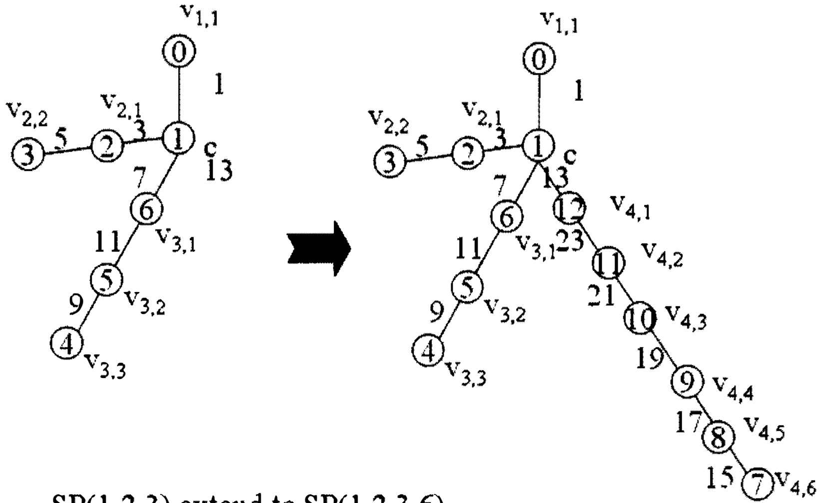
4. Extension and Open Problem.

In this section we want to show some applications of previous results.

Theorem 4.1. Given a (1,2)-SI spider $SP(a_1,a_2,\ldots,a_k)$ with $a_1 \le a_2 \le \ldots \le a_k$, we can extend to a (1,2)-SI spider $SP(a_1,a_2,\ldots,a_k, a_{k+1})$ where $a_{k+1} = x+2-c$, where c is the vertex label of the center vertex of the spider, x+1 is the largest vertex label of the leg a_k.

Proof: Let c be the center vertex of the (1,2)-SI spider $SP(a_1,a_2,\ldots,a_k)$. Assume the leg of length a_k of the spider $SP(a_1,a_2,\ldots,a_k)$ has the highest edge label 2x+1 which has adjacent vertices of labels x and x+1. The vertex label x+1 is the highest vertex label in $SP(a_1,a_2,\ldots,a_k)$. Now we can extend $SP(a_1,a_2,\ldots,a_k)$ by adding another leg of length (x+2-c) in such a way that the vertex adjacent to center vertex have vertex label 2x+3-c which will induce the edge (c, 2x+3-c) with edge label 2x+3, the rest vertices of a_{k+1} from the end vertex have vertex labels (x+2), (x+3), …,(2x+2-c). It is clear we have a (1,2)-SI labeling for $SP(a_1,a_2,\ldots,a_k ,a_{k+1})$.□

Example 13. Figure 18 depicts the way to extend a (1,2)-SI spider SP(1,2,3) to a (1,2)-SI spider SP(1,2,3,6). We see c has label 1, the highest edge label of SP(1,2,3) is 2x+1= 11. Thus the highest vertex label in SP(1,2,3) is x+1= 6. Thus by append a path P_7 of lengthx+2-c =7-1=6 and label the vertices $v_{4,6}$, $v_{4,5}$, $v_{4,4}$, $v_{4,3}$, $v_{4,2}$, $v_{4,1}$ by 7,8,9,10,11,12. We obtain a (1,2)-SI labeling of SP(1,2,3,6).



SP(1,2,3) extend to SP(1,2,3,6)

Figure 18.

One can see that many spiders such as SP(2,2,2,2) is (3,1)-SI , (4,1)-SI and (5,1)-SI but not (1,2)-SI. However, they satisfy the bipartition condition $||M|-|N|| \leq 1$.

We propose here the following open problem.

Problem. Characterize spider T such that T is (1,2)-SI.

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