ON FRIENDLY INDEX SETS OF TREES

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ABSTRACT. For a graph $G = (V, E)$ and a coloring $f : V(G) \to \mathbb{Z}_2$ let $v_f(i) = |f^{-1}(i)|$. f is said to be friendly if $|v_f(1) - v_f(0)| \leq 1$. The coloring $f : V(G) \to \mathbb{Z}_2$ induces an edge labeling $f^* : E(G) \to \mathbb{Z}_2$ defined by $f^*(xy) = f(x) + f(y)$ $\forall xy \in E(G)$, where the summation is done in $\mathbb{Z}_2$. Let $e_f(i) = |{f^*}^{-1}(i)|$. The friendly index set of the graph G, denoted by $FI(G)$, is defined by

$$FI(G) = \{|e_f(1) - e_f(0)| : f \text{ is a friendly vertx labeling of } G\}.$$

In this paper we will determine the friendly index set of certain classes of trees, which in turn will verify the validity of the conjecture that the elements of friendly index set of any tree form an arithmetic progression.

Key Words: coloring, friendly index set, Fibonacci and Lucas trees.
AMS Subject Classification: 05C15, 05C25, 05C78

1. Introduction

Let $G = (V, E)$ be a graph and $f : V(G) \to \mathbb{Z}_2$ a vertex labeling (coloring) of G. For $i \in \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$. The coloring f is said to be *friendly* if $|v_f(1) - v_f(0)| \leq 1$. For example, consider the graph G, illustrated in Figure 1, which has six vertices. To have a friendly coloring, the condition $|v_f(1) - v_f(0)| \leq 1$ implies that three vertices should be marked 0 and the other three 1.

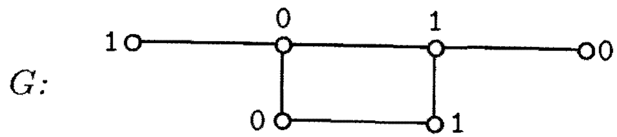

FIGURE 1. A typical friendly labeling of G.

Any vertex labeling $f : V(G) \to \mathbb{Z}_2$ induces an edge labeling $f^* : E(G) \to \mathbb{Z}_2$ defined by $f^*(xy) = f(x) + f(y)\ \forall xy \in E(G)$. For $i \in \mathbb{Z}_2$, let $e_f(i) = |f^{*-1}(i)|$. The number $N(f) = |e_f(1) - e_f(0)|$ is called the *friendly index* of f. The *friendly index set* of the graph G, denoted by $FI(G)$, is defined by

$$FI(G) = \{N(f) : \ f \text{ is a friendly coloring of } G\ \}.$$

A friendly coloring $f : V(G) \to \mathbb{Z}_2$ is called *maximum friendly coloring* of G if its friendly index is the maximum element of $FI(G)$. Note that if $f : V(G) \to \mathbb{Z}_2$ is a friendly coloring, so is its inverse coloring $g : V(G) \to \mathbb{Z}_2$ defined by $g(v) = 1 - f(v)\ \forall v \in V(G)$. Moreover, $N(g) = N(f)$. The induced edge labeling of the graph G, Figure 1, is illustrated in Figure 2. It is easy to see that $FI(G) = \{0, 2, 6\}$.

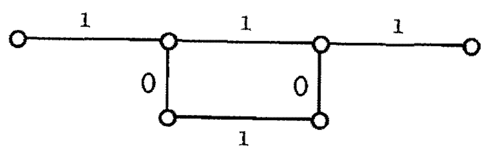

FIGURE 2. The induced edge labeling of G.

Three friendly colorings of G that provide these numbers are presented in Figure 3.

In 1978, Ibrahim Cahit [2, 3, 4] introduced the concept of cordial labeling as weakened version of the less tractable graceful and harmonious labeling. A graph G is said to be cordial if it admits a friendly labeling with index 0 or 1. Cahit, among other facts, proved that

1. every tree is cordial;
2. The complete graph K_n is cordial if and only if $n \leq 3$;
3. The complete bipartite graph $K(m, n)$ is cordial $(m, n \in \mathbb{N})$;
4. The wheel W_n is cordial if and only if $n \not\equiv 3 \pmod 4$;
5. In an Eulerian graph $G = (p, q)$ if $p \equiv 0 \pmod 4$, then it is not cordial.

M. Hovay [8], later generalized the concept of cordial graphs and introduced A-cordial labelings. A graph G is said to be A-cordial if it admits a labeling $f : V(G) \to A$ such that for every $i, j \in A$,

$$|v_f(i) - v_f(j)| \leq 1 \text{ and } |e_f(i) - e_f(j)| \leq 1.$$

Cordial graphs have been studied extensively. Interested readers are referred to a number of relevant literature that are mentioned in the bibliography section, including [1, 5, 7, 9, 11, 13, 17].

In this paper, we will focus on group $A = \mathbb{Z}_2$. The *friendly index set* of a graph G, denoted by $FI(G)$, is defined to be $\{N(f) = |e_f(1) - e_f(0)| :$ f is friendly coloring $\}$. When there is no ambiguity we will drop the subscript f. Note that if 0 or 1 is in $FI(G)$, then G is cordial. Thus the concept of friendly index sets could be viewed as a generalization of cordiality.

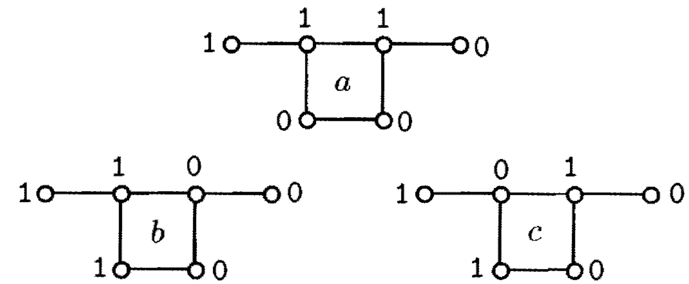

FIGURE 3. $FI(G) = \{0, 2, 6\}$.

In general, the elements of $FI(G)$ do not necessarily form an arithmetics progression. However, Lee and Ng [15] conjectured that the elements of the friendly index set of any tree will form an arithmetic progression. This conjecture has been verified for several classes of trees. To further verify the validity of this conjecture, in this paper, we will find the friendly index sets of several classes of trees. First a few well-known results [14]:

Theorem 1.1. *For any graph G with q edges, $FI(G) \subset \{q - 2i : i = 0, 1, 2, \cdots, \lfloor q/2 \rfloor\}$.*

Theorem 1.2. *Let $1 \leq m \leq n$. For the complete bipartite graph $K_{m,n}$ we have*

$$FI(K_{m,n}) = \begin{cases} \{(m - 2i)^2 : 0 \leq i \leq \lfloor m/2 \rfloor\} & \text{if } \ m + n \text{ is even;} \\ \{i(i+1) : 0 \leq i \leq m\} & \text{if } \ m + n \text{ is odd.} \end{cases}$$

For any $n \geq 1$, the complete bipartite graph $K(1, n)$ is called a *star* and is denoted by $ST(n)$. Stars are the only trees of diameter 2, for which we have:

Corollary 1.3. $FI(ST(n)) = \begin{cases} \{0,2\} & \text{if } n \text{ is even;} \\ \{1\} & \text{if } n \text{ is odd.} \end{cases}$

Theorem 1.4. *The friendly index set of a full binary tree with depth $d > 1$ is $\{0, 2, 4, \cdots, 2^{d+1} - 4\}$.*

2. Trees with perfect matching

Definition 2.1. A *matching* in a graph is a set of edges with no shared endpoints. A matching M in a graph G is said to be a *perfect matching* if every vertex of G is incident with an edge in M.

Note that every graph with perfect matching has even number of vertices. Also, stars $ST(n)$ $(n \geq 2)$ are among trees that do not have a perfect matching. As a result, if we identify one of the end-vertices of $ST(n)$ $(n \geq 3)$ with a vertex of any graph G, the resulting graph does not have a perfect matching.

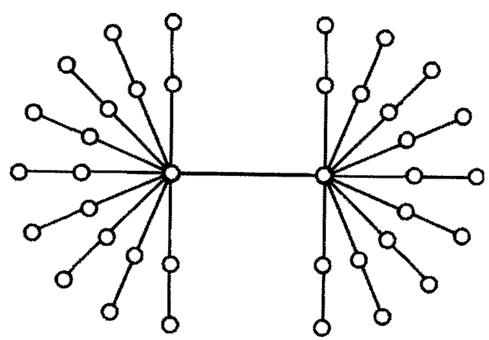

FIGURE 4. An example of a tree with perfect matching.

Observation 2.2. *Every terminal edge of a tree with perfect matching M is in M.*

Observation 2.3. *A tree T $(|T| \geq 3)$ with perfect matching has at least one P_3, the path of order 3, pendant. In fact, the two end portion of the longest path of T would have P_3 pendants.*

Theorem 2.4. *If $T = (p, q)$ is a tree with perfect matching, then $FI(T) = \{1, 3, 5, \cdots, q\}$.*

Proof. Let M be a perfect matching of the tree T. We will proceed by induction on $|M|$. Clearly, the theorem is true for $|M| = 1$, 2. Assume the theorem is true whenever the the perfect matching has n elements. Let T be a tree with a perfect matching M such that $|M| = n + 1$. This implies that $|E(T)| = 2|M| + 1$. By observation 2.3, T has at least a P_3 pendant. Let $u \sim v \sim w$ be such a pendant with vw being the terminal edge. Consider $T' = T - \{v, w\}$, which is a tree with perfect matching. In fact, $M - vw$ is a perfect matching in T' with $|M - vw| = n$. By the induction hypothesis $FI(T') = \{1, 3, \cdots, 2n - 1\}$.

Let $f : V(T') \to \{0, 1\}$ be an arbitrary friendly coloring of T' and define $\phi : V(T) \to \{0 1\}$ by

$$\phi(x) = \begin{cases} f(x) & \text{if} \quad x \neq v, w; \\ f(u) & \text{if} \quad x = v; \\ 1 - f(u) & \text{if} \quad x = w. \end{cases}$$

Then ϕ is a friendly coloring of T with $e_\phi(1) = 1 + e_f(1)$ and $e_\phi(0) = 1 + e_f(0)$. Therefore,

$$|e_\phi(1) - e_\phi(0)| = |e_f(1) - e_f(0)|,$$

which implies that $FI(T') = \{1, 3, \cdots, 2n - 1\} \subset FI(T)$.

Now, let $g : V(T') \to \{0, 1\}$ be a maximum friendly coloring of T' with $e_g(1) = 2n - 1$ (or $e_g(0) = 0$) and define $\psi : V(T) \to \{0 1\}$ by

$$\psi(x) = \begin{cases} f(x) & \text{if} \quad x \neq v, w; \\ 1 - f(u) & \text{if} \quad x = v; \\ f(u) & \text{if} \quad x = w. \end{cases}$$

Then ψ is a friendly coloring of T with $e_\psi(1) = 2 + e_g(1) = 2n + 1$, which implies that $2n + 1 \in FI(T)$. □

With a similar argument presented in the proof of theorem 2.4 one can show that

Theorem 2.5. $FI(P_n) = \{n - 1 - 2i : 0 \leq i \leq \lfloor \frac{n-1}{2} \rfloor\}$, *where P_n $(n \geq 2)$ is the path of order n.*

The theorem 2.5 implies that for any natural number n there is a connected graph G such that $FI(G) = \{n - 2i : i = 1, 2, \cdots, \lfloor n/2 \rfloor\}$, which is an arithmetic progression with common difference being 2.

Problem 2.6. *Given any even integer $d > 2$, find a connected graph whose friendly index set forms an arithmetic progression with common difference d.*

Frucht and Harary [6] introduce the corona of two graphs G and H, denoted by $G©H$, to be the graph with base G such that each vertex $v \in V(G)$ is joined to all vertices of a separate copy of H.

Corollary 2.7. *Given a tree T with q edges, the coronation $T©K_1$ is a tree with perfect matching and its friendly index set is $\{1, 3, 5, \cdots, 2q + 1\}$.*

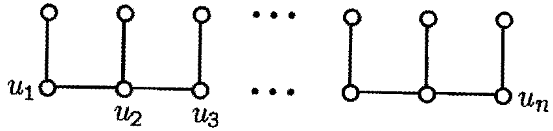

FIGURE 5. $FI(P_n \copyright K_1) = \{1, 3, \cdots, 2n-1\}$.

3. Fibonacci and Lucas Trees

Fibonacci Trees, denoted by FT_n, are defined inductively as follows: FT_1 is the trivial tree with one vertex, FT_2 is the path P_2, and for $n \geq 3$, $FT_n = (V_n, E_n)$ is the binary tree of root r_n, whose left and right children are FT_{n-1} and FT_{n-2}, respectively.

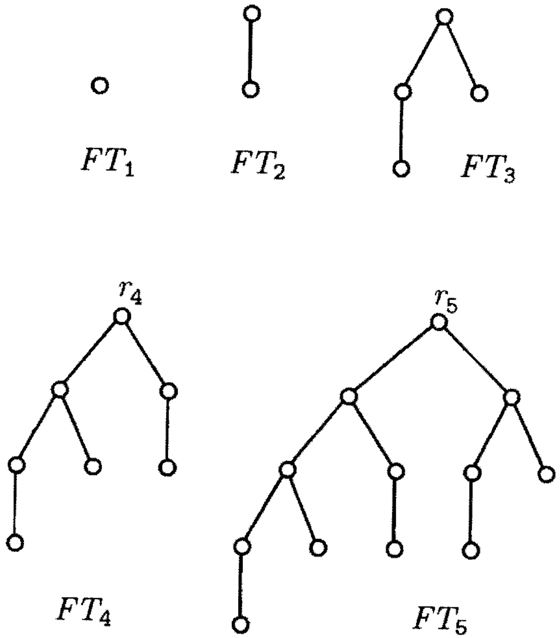

FIGURE 6. Graphs of the first five Fibonacci trees.

Observation 3.1. *Let $p_n = |V_n|$, be the number of vertices of FT_n. Then $p_n = A_{n+2} - 1$, where A_n is the n^{th} Fibonacci number.*

Theorem 3.2. *For $n \geq 3$, the friendly index set of FT_n is $\{\,|E_n| - 2i : i = 0, 1, 2, \cdots, \lfloor q_n/2 \rfloor\}$.*

Moreover, every element of the index set can be obtained by a friendly coloring $f : V_n \to \mathbb{Z}_2$ with $N(f) = e_f(1) - e_f(0)$, and the color of root is 1.

Proof. We proceed by induction on n. Clearly the theorem is true for $n = 2, 3, 4$. For FT_4, it is illustrated in Figure 7. Suppose it is true for all k with $3 \leq k \leq n$ and consider FT_{n+1}. Let $b_1 \in \{\,|E_n| - 2i : i = 1, 2, \cdots, \lfloor q_n/2 \rfloor\}$ and $b_2 \in \{\,|E_{n-1}| - 2i : i = 1, 2, \cdots, \lfloor q_{n-1}/2 \rfloor\}$. Also, let $\phi : V_n \to \mathbb{Z}_2$ be a friendly coloring of FT_n with $e_\phi(1) - e_\phi(0) = b_1$ and $\psi : V_{n-1} \to \mathbb{Z}_2$ be a friendly coloring of FT_{n-1} with $e_\psi(1) - e_\psi(0) = b_2$. Without loss of generality, by considering the inverse coloring if necessary, we may assume that the roots of FT_n and FT_{n-1} are labeled 1 or 0.

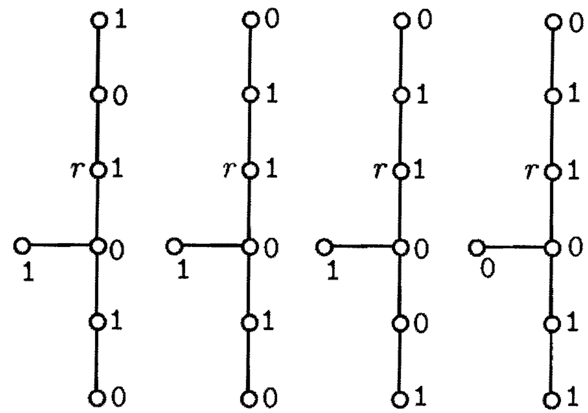

FIGURE 7. Four desired friendly colorings of FT_4.

Define the function $f : V_{n+1} \to \mathbb{Z}_2$ by

$$f(x) = \begin{cases} \phi(x) & \text{if} \quad x \in V_n; \\ \psi(x) & \text{if} \quad x \in V_{n-1}; \\ j & \text{if} \quad x = r_{n+1}, \end{cases}$$

where $j \in \mathbb{Z}_2$. To make sure that f is a friendly coloring we will consider two cases:

Case A. $|V_n|$ and $|V_{n-1}|$ are even. By defining $f(r_{n+1}) = 1$, f would be a friendly coloring of FT_{n+1}. If the roots of FT_n and FT_{n-1} are labeled 0, then $N(f) = b_1 + b_2 + 2$. If the roots of FT_n and FT_{n-1} are labeled differently, then $N(f) = b_1 + b_2$. Hence, $b_1 + b_2$, $b_1 + b_2 + 2$ are in the friendly index set of FT_{n+1} and these numbers will completely determine the set.

Case B. One of the numbers $|V_n|$ or $|V_{n-1}|$ is odd and the other one is even. Assume that $|V_n|$ is odd, $v_\phi(1) = v_\phi(0) + 1$, and the root of FT_n is labeled 1. This implies that $f(r_{n+1}) = 0$. If the root of FT_{n-1} is labeled 1, then $N(f) = b_1 + b_2 + 2$. If the root of FT_{n-1} is labeled 0, then $N(f) = b_1 + b_2$. Hence, $b_1 + b_2$, $b_1 + b_2 + 2$ are in the friendly index set of FT_{n+1} and these numbers will completely determine this set. Finally, we consider the inverse coloring of f, if needed, to guarantee that $f(r_{n+1}) = 1$. □

Definition 3.3. *Lucas Trees, denoted by LT_n, are defined inductively as follows: LT_1 is the trivial tree with one vertex, LT_2 is the path P_3 with the middle vertex being its root, and for $n \geq 3$, $LT_n = (V_n, E_n)$ is the binary tree of root r_n, whose left and right children are LT_{n-1} and LT_{n-2}, respectively.*

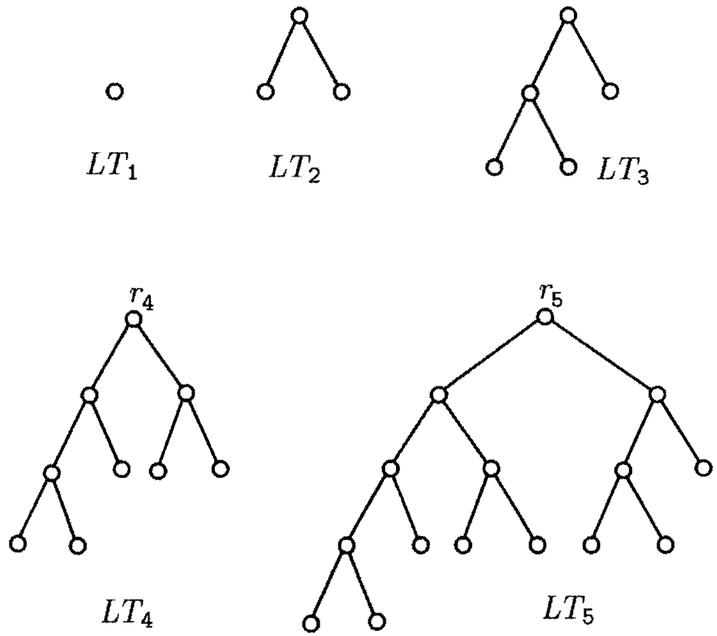

FIGURE 8. Graphs of the first five Lucas trees.

Observation 3.4. *Let $p_n = |V_n|$, be the number of vertices of LT_n. Then $p_n = 2A_{n+1} - 1$, where A_n is the n^{th} Fibonacci number.*

Theorem 3.5. *For $n \geq 3$, the friendly index set of the Lucas tree $LT_n = (V_n, E_n)$ is $\{0, 2, 4, \cdots, |E_n|\}$.*
Moreover, every element b of this index set can be obtained by a friendly coloring $f : V_n \to \mathbb{Z}_2$ with the following properties:

(a) $f(r_n) = 1$;
(b) $b = N(f) = e_f(1) - e_f(0)$;
(c) *If* $b \equiv 0 \pmod 4$, *then* $v_f(1) = v_f(0) + 1$; *and*
(d) *If* $b \equiv 2 \pmod 4$, *then* $v_f(0) = v_f(1) + 1$.

Proof. We proceed by induction on n. Clearly the theorem is true for $n = 2, 3, 4$. For LT_4, it is illustrated in Figure 9. Suppose it is true for all k with $3 \leq k \leq n$ and consider LT_{n+1}. Let $b_1 \in \{0, 2, 4, \cdots, |E_n|\}$ and $b_2 \in \{0, 2, 4, \cdots, |E_{n-1}|\}$. We wish to show that $b_1 + b_2$ and $b_1 + b_2 + 2$ belong to the index set of LT_{n+1}.

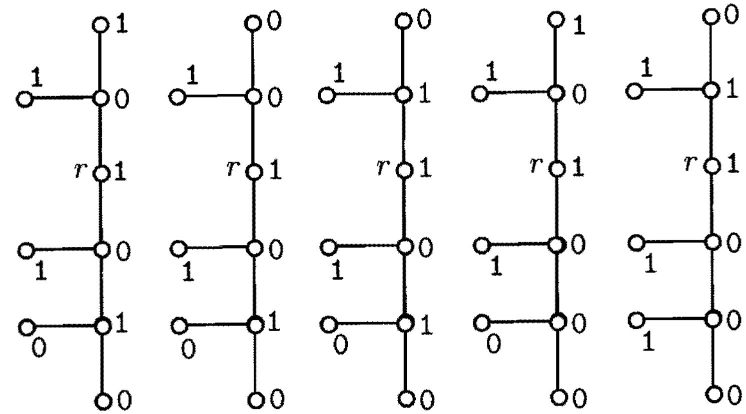

FIGURE 9. Five desired friendly colorings of LT_4.

Using the induction hypothesis, let $\phi : V_n \to \mathbb{Z}_2$ be a friendly coloring of FT_n with $b_1 = e_\phi(1) - e_\phi(0)$ and $\psi : V_{n-1} \to \mathbb{Z}_2$ be a friendly coloring of FT_{n-1} with $b_2 = e_\psi(1) - e_\psi(0)$. Also, let $\phi(r_n) = \psi(r_{n-1}) = 1$.
First, we wish to prove that $0 \in FI(LT_{n+1})$. Choose $b_1 = 0$ and $b_2 = 2$. Then $v_\phi(1) = v_\phi(0) + 1$, and $v_\psi(0) = v_\psi(1) + 1$. This situation is summerized in Figure 10.
Define the function $f : V_{n+1} \to \mathbb{Z}_2$ by

$$f(x) = \begin{cases} \phi(x) & \text{if} \quad x \in V_n; \\ \psi(x) & \text{if} \quad x \in V_{n-1}; \\ 1 & \text{if} \quad x = r_{n+1}, \end{cases}$$

which is a friendly coloring of LT_{n+1}, with $N(f) = b_1 + b_2 - 2 = 0$.
Next, assume b_1 or $b_2 \neq 0$; for example let $b_1 \neq 0$. Also, let $\lambda : V_n \to \mathbb{Z}_2$ be a friendly coloring of FT_n with $b_1 - 2 = e_\lambda(1) - e_\lambda(0)$. Depending on whether b_1 or b_2 are multiple of 4 or not, we will consider the following two cases:

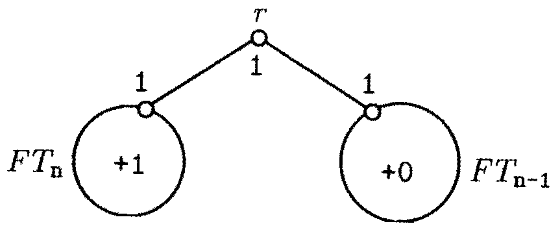

FIGURE 10. $+i$ refers to extra i in the coloring of the graph.

Case 1. At least one of b_1 or b_2 is a multiple of 4; assume $b_2 \equiv 0 \pmod 4$. In this case $v_\psi(1) = v_\psi(0) + 1$. The function $g : V_{n+1} \to \mathbb{Z}_2$ defined by

$$g(x) = \begin{cases} \phi(x) & \text{if} \quad x \in V_n; \\ \psi(x) & \text{if} \quad x \in V_{n-1}; \\ 0 & \text{if} \quad x = r_{n+1}, \end{cases}$$

is a friendly coloring of LT_{n+1}, with $N(g) = b_1 + b_2 + 2$. Also, the function $h : V_{n+1} \to \mathbb{Z}_2$ defined by

$$h(x) = \begin{cases} \lambda(x) & \text{if} \quad x \in V_n; \\ \psi(x) & \text{if} \quad x \in V_{n-1}; \\ 0 & \text{if} \quad x = r_{n+1}, \end{cases}$$

is a friendly coloring of LT_{n+1}, with $N(h) = b_1 + b_2$. The inverse coloring of g, h will satisfy the conditions of the theorem.

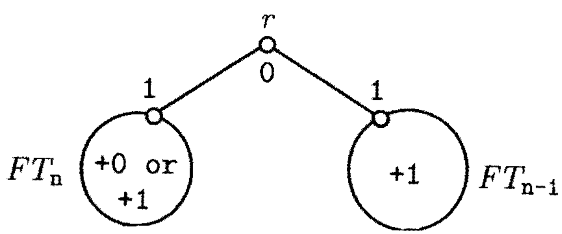

FIGURE 11. $b_2 \equiv 0 \pmod 4$.

Case 2. $b_1, b_2 \equiv 2 \pmod 4$. In this case $b_1 - 2$ is multiple of 4 and the function $h : V_{n+1} \to \mathbb{Z}_2$, defined above, is a friendly coloring of LT_{n+1}, with $N(h) = b_1 + b_2$. Again, the inverse coloring of h, will satisfy the conditions of the theorem.
This proves that if b_1 or $b_2 \neq 0$, then $b_1 + b_2$ and $b_1 + b_2 + 2$ are members of $FI(LT_{n+1})$. □

We conclude the paper with the improved version of the conjecture:

Conjecture 3.6. *The elements of the friendly index set of any tree form an arithmetic progression with common difference* 2.

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