On Friendly Index Sets of Broken Wheels with Three Spokes

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Abstract

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and let A be an abelian group. A labeling $f : V(G) \to A$ induces a edge labeling $f^* : E(G) \to A$ defined by $f^*(xy) = f(x) + f(y)$ for each $xy \in E$. For each $i \in A$, let $v_f(i) = card\{v \in V(G) : f(v) = i\}$ and $e_f(i) = card\{e \in E(G) : f^*(e) = i\}$. Let $c(f) = \{|e_f(i) - e_f(j)| : (i,j) \in A \times A\}$. A labeling f of a graph G is said to be A-friendly if $|v_f(i) - v_f(j)| \leq 1$ for all $(i,j) \in A \times A$. If $c(f)$ is a $(0,1)$-matrix for an A-friendly labeling f, then f is said to be A-cordial. When $A = \mathbb{Z}_2$, the friendly index set of the graph G, $\mathrm{FI}(G)$, is defined as $\{|e_f(0) - e_f(1)| :$ the vertex labeling f is $\mathbb{Z}_2$-friendly$\}$. In [15] the friendly index set of a cycle is completely determined. We consider the friendly index sets of broken wheels with three spokes.

Keywords and phrases: vertex labeling, friendly labeling, cordiality, friendly index set, cycle, broken wheels with three spokes.

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1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and let A be an abelian group. A labeling $f : V(G) \to A$ induces an edge labeling $f^* : E(G) \to A$ defined by $f^*(xy) = f(x) + f(y)$ for each $xy \in E(G)$. For each $i \in A$, let $v_f(i) = card\{v \in V(G) : f(v) = i\}$ and $e_f(i) = card\{e \in E(G) : f^*(e) = i\}$. Let $c(f) = \{|e_f(i) - e_f(j)| : (i,j) \in A \times A\}$. A labeling f of a graph G is said to be A-friendly if $|v_f(i) - v_f(j)| \leq 1$ for all $(i,j) \in A \times A$. If $c(f)$ is a $(0,1)$-matrix for an A-friendly labeling f, then f is said to be A-cordial.

The notion of A-cordial labelings was first introduced by Hovey [10], who generalized the concept of cordial graphs of Cahit [2, 3]. Cahit considered $A = Z_2$ and he proved the following: every tree is cordial; K_n is cordial if and only if $n \leq 3$; $K_{m,n}$ is cordial for all m and n; the wheel W_n is cordial if and only if $n \not\equiv 3 \pmod 4$; C_n is cordial if and only if $n \not\equiv 2 \pmod 4$; and an Eulerian graph is not cordial if its size is $2 \pmod 4$. Benson and Lee [1] showed a large class of cordial regular windmill graphs that include the friendship graphs as a subclass.

Lee and Liu [14] investigated cordial complete k-partite graphs. Kuo, Chang and Kwong [13] determined all m and n for which mK_n is cordial. Ho, Lee and Shee investigated the construction of cordial graphs by Cartesian product and composition [8], and completely characterized cordial generalized Petersen graphs [9]. Seoud and Maqsoud [23] proved that certain cylinder graphs are cordial. Several constructions of cordial graphs were considered in [20, 22, 24, 25]. For more details of known results and open problems on cordial graphs, see [4, 7].

In this paper, we will exclusively focus on $A = Z_2$, and drop the reference to the group. When the context is clear, we will also drop the subscript f. In [6] the following concept was introduced.

Definition 1. The friendly index set FI(G) of a graph G is defined as $\{|e_f(0) - e_f(1)| :$ the vertex labeling f is friendly$\}$.

Note that if 0 or 1 is in FI(G), then G is cordial. Thus the concept of friendly index sets could be viewed as a generalization of cordiality. Cairnie and Edwards [5] have determined the computational complexity of cordial labeling and Z_k-cordial labeling. They proved that to decide whether a graph admits a cordial labeling is NP-complete. Even the restricted problem of deciding whether a connected graph of diameter 2 has a cordial labeling is NP-complete. Thus in general it is difficult to determine the friendly index sets of graphs.

In [11, 12, 15, 16, 17, 18, 19, 21] the friendly index sets of a few classes of graphs are determined. The following result was established.

Theorem 1.1 *For any graph with q edges, the friendly index set FI(G) $\subseteq \{0, 2, 4, \cdots, q\}$ if q is even, and FI(G) $\subseteq \{1, 3, \cdots, q\}$ if q is odd.*

Example 1. Figure 1 illustrates the friendly index set of the cycle C_8 with two parallel chords FI($PC(8,2)$) $= \{6, 4, 2, 0\}$.

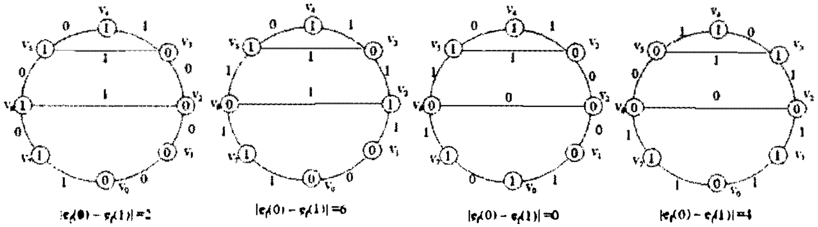


Figure 1: FI($PC(8,2)$)

Example 2. FI($K_{3,3}$) $= \{1, 9\}$ and FI($C_3 \times K_2$) $= \{1, 3, 5\}$.

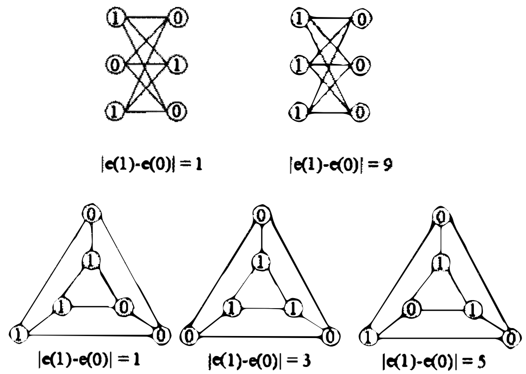


Figure 2: FI($K_{3,3}$) and FI($C_3 \times K_2$)

Example 3. Figure 3 shows that FI($C_4 \times_L St(3)$) $= \{0, 4, 8, 12, 16\}$.

Lee and Ng [15] showed that

Theorem 1.2 *The friendly index set of a cycle is given as follows:*

1. $FI(C_{2n}) = \begin{cases} \{0, 4, 8, \cdots, 2n\} & \text{if } n \text{ is even,} \\ \{2, 6, 10, \cdots, 2n\} & \text{if } n \text{ is odd.} \end{cases}$

2. $FI(C_{2n+1}) = \{1, 3, 5, \cdots, 2n-1\}$

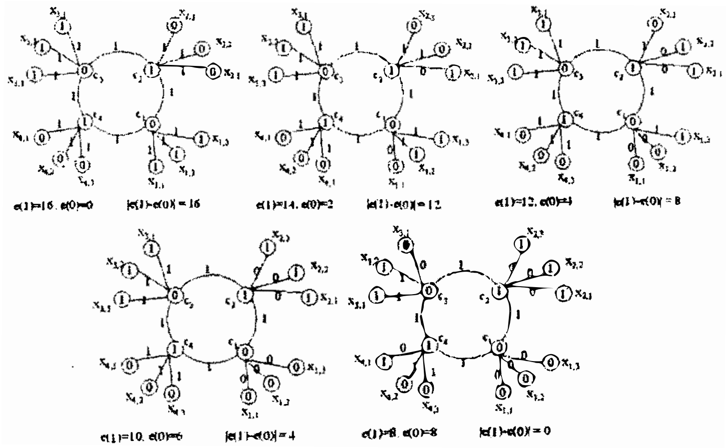


Figure 3: $\mathrm{FI}(C_4 \times_L St(3))$

We observe the same phenomenon for cycles with parallel chords. In [18], Lee and Ng showed that for a cycle with an arbitrary non-empty set of parallel chords, the values in its friendly index set form an arithmetic progression with common difference 2. If the chords are not parallel, the numbers in the friendly index set might not form an arithmetic progression. See [17] on the friendly index sets of Möbius ladders. The purpose of this paper is to investigate the friendly index sets of the following graphs.

Definition 2. For integers a, b, c with $1 \le a \le b \le c$, the broken wheel $W(a,b,c)$ with three spokes is the graph constructed from a complete graph K_4 where $V(K_4) = \{u_1, u_2, u_3, c\}$ by insert $(a-1)$ vertices $\{x_{1,1}, x_{1,2}, \cdots, x_{1,a-1}\}$ along the edge (u_1, u_2), $(b-1)$ vertices $\{x_{2,1}, x_{2,2}, \cdots, x_{2,b-1}\}$ along the edge (u_2, u_3) and $(c-1)$ vertices $\{x_{3,1}, x_{3,2}, \cdots, x_{3,c-1}\}$ along the edge (u_3, u_1).

Example 4. Figure 4 depicts the broken wheels with three spokes $W(2,2,3)$ and $W(4,4,4)$.

In this paper, we investigate when the numbers in $\mathrm{FI}(G)$ for a broken wheel with three spokes $W(a,b,c)$ form an arithmetic progression. We will use the terms "cycle" and "triangle" interchangeably, depending on which one is clearer geometrically.

Theorem 1.3 *The friendly index set of a cycle is given as follows:*

1. *If all of a, b and c are even, then* $\max\{FI(W(a,b,c))\} = a+b+c+3$.

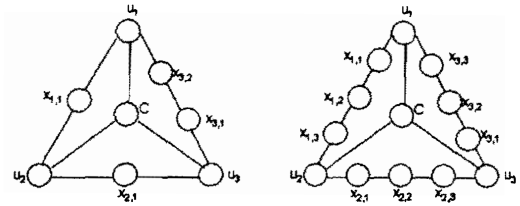


Figure 4: $W(2,2,3)$ and $W(4,4,4)$

2. *If exactly two of a, b and c are even, then* $\max\{FI(W(a,b,c))\} = a+b+c+1$.
3. *If exactly one of a, b and c is even, then* $\max\{FI(W(a,b,c))\} = a+b+c+1$.
4. *If all of a, b and c are odd, then* $\max\{FI(W(a,b,c))\} = a+b+c-1$.

Proof. The graph $W(a,b,c)$ has $(a+b+c+3)$ edges. Obviously it is impossible for a friendly vertex labeling to have all induced edge labels equal to 0, i.e., $e(1) \neq 0$.

1. Label the vertices of the cycle C_{a+b+c} alternately with 0s and 1s. All the edges of C_{a+b+c} have induced label 1.

   The vertex labels at the three corners of the triangle are the same. Choose the complementary label for the center of $W(a,b,c)$, and all edges of $W(a,b,c)$ have induced label 1.

2. The cycle C_{a+b+c} is an odd cycle. By a result in [15], there must be at least one 0-edge in the cycle. Thus $\max\{\mathrm{FI}(W(a,b,c))\} \le a+b+c+1$.

   Without loss of generality, assume that a and b are even while c is odd. Label the vertices of the triangle, starting with the side with a edges, followed by the side with b edges, and finally the side with c edges, alternately with 0's and 1's. At the end, there will be two 0-vertices adjacent to each other, giving exactly one 0-edge in the cycle. Note that all corner vertices have label 0. Label the center 1, and all spokes will have induced label 1. Thus $e(1) - e(0) = a+b+c+1$.

3. If $e(0) = 0$, the vertices of the cycle C_{a+b+c} must have alternate labels of 0's and 1's. The three corner vertices cannot all have the same label. Then not all spokes can have induced label 1, contradicting $e(0) = 0$. Thus $\max\{\mathrm{FI}(W(a,b,c))\} \le a+b+c+1$.

To show that this is attainable, again label the vertices of the cycle C_{a+b+c} alternately with 0's and 1's. All edge of the cycle have induced label 1. Note that two corner vertices have the same label, while the remaining corner vertex has the complementary label. Use this complementary label for the center. Then two spokes have induced label 1, while the remaining spoke has induced label 0. Thus $e(1) - e(0) = a + b + c + 1$.

4. As in part (b), $\max\{\text{FI}(W(a,b,c))\} \leq a + b + c + 1$.

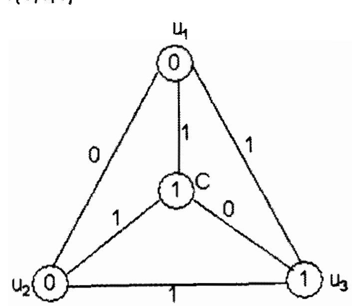
   If $\max\{\text{FI}(W(a,b,c))\} = a+b+c+1$, either $e(1) = 1$ or $e(0) = 1$. In the first case, the sole 1-edge must be a spoke, requiring all the vertex labels of the cycle be the same. But this cannot be a friendly vertex labeling. In the second case, the sole 0-edge must be in the cycle, and all three spokes must have induced label 1. Thus all but 1 pair of adjacent vertices of the cycle must have complementary vertex labels. Thus not all corner vertices can have the same label, forcing at least one spoke to have an induced edge label of 0, which is a contradiction.

   It suffices to show that $e(1) - e(0) = a+b+c-1$ is attainable. Label the vertices of the cycle alternately with 0's and 1's. At the end, there will be two 0-vertices adjacent to each other, giving exactly one 0-edge on the cycle. Note that two corner vertices have label 0, and one corner vertex has label 1. Label the center 1, exactly one spoke will have induced label 0. Thus $e(1) - e(0) = a + b + c - 1$.

□

2 Friendly index set of broken wheel $W(a,b,c)$, where a,b,c are all odd

W(1,1,1)



|e(0)-e(1)| = 2

Obviously $\text{FI}(W(1,1,1)) = \{2\}$. From now on, we disregard this case unless otherwise specified. We also note that using complementary vertex labels will not change the values of $e(0)$ and $e(1)$. Finally we let the vertices of P_{n+1} be $v_1, v_2, \cdots, v_{n+1}$.

Lemma 2.1 *Let n be odd. In P_{n+1}, it is possible for $e(0) = 0, 2, \cdots, M$, where $M = \frac{n-1}{2}$ if $n \equiv 1 \pmod 4$, and $M = \frac{n+1}{2}$ if $n \equiv 3 \pmod 4$, with $v(0) = v(1)$, and with complementary labels for the two end-vertices.*

Proof. Label the vertices of P_{n+1} from v_1 to v_{n+1} alternately with 0's and 1's in this order. Obviously $e(0) = 0$ here.

First consider $n \equiv 1 \pmod 4$. Divide the vertices $v_2, \cdots, v_n$ into $\frac{n-1}{4}$ blocks of 4, each with vertex labels $1, 0, 1, 0$. Start with the block v_2, v_3, v_4, v_5 and change the vertex labels to $0, 1, 1, 0$ respectively. Then $e(0) = 2$. Continue with the rest of the blocks, and $e(0)$ finally reaches $\frac{n-1}{2}$.

Next consider $n \equiv 3 \pmod 4$. Divide the vertices $v_2, \cdots, v_n$ into $\frac{n-3}{4}$ blocks of 4, with a final block of the 2 vertices v_{n-1} and v_n. Perform exactly the same procedure as in the last case to the blocks of 4. Finally interchange the vertex labels of v_{n-1} and v_n to achieve $e(0) = \frac{n-3}{2} + 2 = \frac{n+1}{2}$. □

Lemma 2.2 *Let n be odd. In P_{n+1}, it is possible for $e(0) = 1, 3, \cdots, M$, where $M = \frac{n+1}{2}$ if $n \equiv 1 \pmod 4$, and $M = \frac{n-1}{2}$ if $n \equiv 3 \pmod 4$, with $v(0) = \frac{n+3}{2}$ and $v(1) = \frac{n-1}{2}$, and with the two end-vertices having the same label.*

Proof. Label the vertices $v_1, v_2, \cdots, v_{n-1}$ with alternate labels 0's and 1's in this order, and the last two vertices v_n and v_{n+1} with 0. Obviously $e(0) = 1$ here.

Note that the subpath P_{n-1} containing $v_1, v_2, \cdots, v_{n-1}$ has $v(0) = v(1)$ with the two end-vertices having complementary labels, and we can apply Lemma 2.1 to this subpath. When $n \equiv 1 \pmod 4$, $n - 2 \equiv 3 \pmod 4$, and $e(0)$ for this subpath can take values from 0 to $\frac{n-1}{2}$. Thus $e(0)$ for the entire path P_{n+1} can take values from 1 to $\frac{n+1}{2}$. When $n \equiv 3 \pmod 4$, $n - 2 \equiv 1 \pmod 4$, and $e(0)$ for this subpath can take values from 0 to $\frac{n-3}{2}$. Thus $e(0)$ for the entire path P_{n+1} can take values from 1 to $\frac{n-1}{2}$. □

Lemma 2.3 *Let n be odd. In P_{n+1}, it is possible for $e(0) = 1, 3, \cdots, M$, where $M = \frac{n+1}{2}$ if $n \equiv 1 \pmod 4$, and $M = \frac{n-1}{2}$ if $n \equiv 3 \pmod 4$, with $v(0) = v(1)$, and with the two end-vertices having the same label.*

Proof. Label the vertices $v_1, v_2, \cdots, v_{n-1}$ with alternate labels 0's and 1's in this order, and the last two vertices v_n and v_{n+1} with 1 and 0 respectively. Obviously $e(0) = 1$ here.

Again we can apply Lemma 2.1 to the subpath P_{n-1} containing $v_1, v_2, \cdots, v$ Essentially the same argument as in the proof of Lemma 2.2 establishes the result. □

Lemma 2.4 *Let a and $b \equiv 1 \pmod 4$, and $c \equiv 3 \pmod 4$. Then $0 \in$ FI($W(a,b,c)$).*

Proof. Label both end-vertices of the path P_{a+1} with 1. Label the other end-vertex of the path P_{b+1} with 0. Label the center with 0. Then two spokes have induced label 1 and the remaining spoke has induced label 0.

Divide the remaining vertices of P_{a+1} into blocks of 4, and label each block using $0, 0, 1, 1$ in this order. Divide the remaining vertices of P_{b+1} into blocks of 4, and label each block using $0, 0, 1, 1$ in this order. Divide the remaining vertices of P_{c+1} into blocks of 4 and a final block of 2, and label each block of 4 using $1, 1, 0, 0$ in this order, and the last two vertices with 0 and 1. This vertex labeling is friendly.

In P_{a+1}, there are $\frac{a+1}{2}$ edges with induced label 0, and $\frac{a-1}{2}$ edges with induced label 1. In P_{b+1}, there are $\frac{b-1}{2}$ edges with induced label 0, and $\frac{b+1}{2}$ edges with induced label 1. In P_{c+1}, there are $\frac{c+1}{2}$ edges with induced label 0, and $\frac{c-1}{2}$ edges with induced label 1. For the entire graph, $e(0) = e(1) = \frac{a+b+c+3}{2}$. □

Lemma 2.5 *Consider $W(a,b,c)$ where all of a, b and c are odd. Then FI($W(a,b,c)$) $= \{0, 2, \cdots, a+b+c-1\}$, except possibly when a and $b \equiv 1 \pmod 4$, and $c \equiv 3 \pmod 4$.*

Proof. By Theorems 1.1 and 1.3, it suffices to show that these values are attainable.

Use the labeling in the proof of Theorem 1.3 to obtain $e(1) = a+b+c+1$ and $e(0) = 2$. For the path P_{a+1}, use Lemma 2.1 to change $e(0)$ in increments to 2 up to the maximum value. Do the same to the path P_{b+1}. Then apply Lemma 2.2 to the path P_{c+1} to change $e(0)$ in increments to 2 up to the maximum value. Recall that exactly one spoke has induced label 0. Thus for the last labeling, $e(0) \geq \frac{a-1}{2} + \frac{b-1}{2} + \frac{c-1}{2} + 1 = \frac{a+b+c-1}{2}$. Then $e(1) \leq \frac{a+b+c+7}{2}$, and so $e(1) - e(0)$ changes from $(a+b+c-1)$ to a value ≤ 4 in decrements of 4. The value 4 is attained only when a and $b \equiv 1 \pmod 4$, and $c \equiv 3 \pmod 4$.

Again use the labeling in the proof of Theorem 1.3, except that the 0-non-corner vertex that is adjacent to a 0-corner vertex has its label interchanged with the center vertex. This can be done because we assume that the graph is not $W(1,1,1)$. Then exactly two spokes have induced label 0 and $e(0) = 3$, $e(1) - e(0) = a+b+c-3$. For the path P_{a+1}, use Lemma 2.1 to change $e(0)$ in increments to 2 up to the maximum value. Do the same to the path P_{b+1}. Then apply Lemma 2.3 to the path P_{c+1} to change $e(0)$ in increments to 2 up to the maximum value. Thus for the last labeling, $e(0) \geq \frac{a-1}{2} + \frac{b-1}{2} + \frac{c-1}{2} + 2 = \frac{a+b+c+1}{2}$. Then $e(1) \leq \frac{a+b+c+5}{2}$, and so $e(1) - e(0)$ changes from $(a+b+c-3)$ to a value ≤ 2 in decrements of 4.

Hence all positive even numbers up to $(a+b+c-1)$ are in FI($W(a,b,c)$). Furthermore 0 is in the friendly index set except possibly when a and $b \equiv 1 \pmod 4$, and $c \equiv 3 \pmod 4$. □

Theorem 2.6 *Consider $W(a,b,c)$ where all of a, b and c are odd. Then FI($W(a,b,c)$) $= \{2\}$ if $a = b = c = 1$, and FI($W(a,b,c)$) $= \{0, 2, \cdots, a+b+c-1\}$ otherwise.*

Proof. The only value that possibly escapes Lemma 2.5 is 0 in the case when a and $b \equiv 1 \pmod 4$, and $c \equiv 3 \pmod 4$. But Lemma 2.4 takes care of this. □

Example 5. Figure 5 shows that FI($W(3,3,3)$) $= \{0, 2, 4, 6, 8\}$.

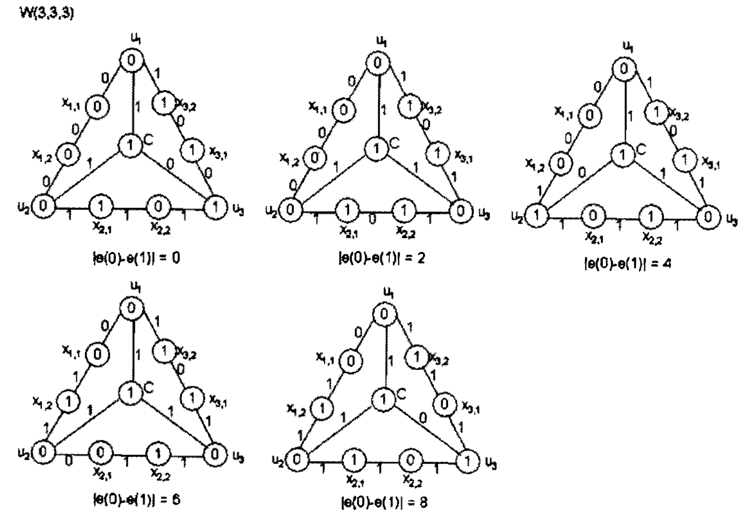


Figure 5: FI($W(3,3,3)$)

Example 6. Figure 6 shows that FI($W(1,3,5)$) $= \{0, 2, 4, 6, 8\}$.

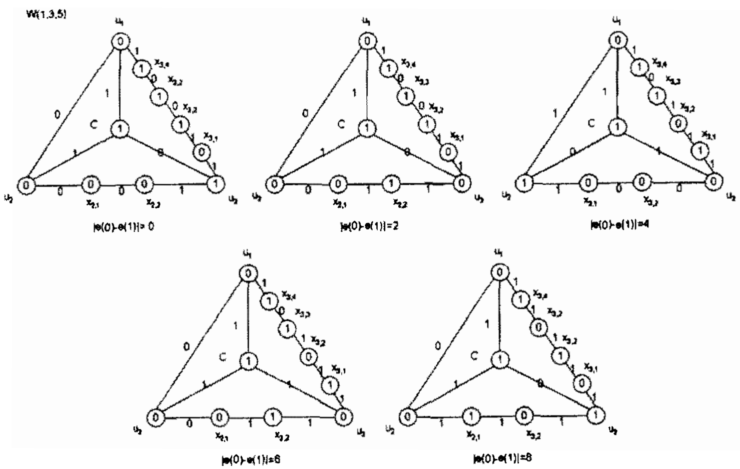


Figure 6: FI($W(1,3,5)$)

3 Friendly index set of broken wheel $W(a,b,c)$, where a,b,c are all even

Lemma 3.1 *Let n be even. In P_{n+1}, it is possible for $e(0) = 0, 2, \cdots, M$, where $M = \frac{n}{2}$ if $n \equiv 0 \pmod 4$, $M = \frac{n+2}{2}$ if $n > 2$ and $n \equiv 2 \pmod 4$, and $M = 0$ if $n = 2$, with $v(0) - v(1) = 1$ and with both end-vertices labeled 0.*

Proof. Label the vertices of P_{n+1} from v_1 to v_{n+1} alternately with 0's and 1's in this order. Obviously $v(0) - v(1) = 1$, and $e(0) = 0$ here. We may now assume that $n > 2$.

First consider $n \equiv 0 \pmod 4$. Divide the vertices $v_2, \cdots, v_{n+1}$ into $\frac{n}{4}$ blocks of 4, each with vertex labels $1, 0, 1, 0$. Start with the block v_2, v_3, v_4, v_5 and change the vertex labels to $0, 1, 1, 0$ respectively. Then $e(0) = 2$. Continue with the rest of the blocks, and $e(0)$ finally reaches $\frac{n}{2}$.

Next consider $n \equiv 2 \pmod 4$. Divide the vertices $v_2, \cdots, v_{n-1}$ into $\frac{n-2}{4}$ blocks of 4. Perform exactly the same procedure as in the last case to these blocks of 4 to reach $e(0) = \frac{n-2}{2}$. Finally interchange the vertex labels of v_{n-1} and v_n to achieve $e(0) = \frac{n-2}{2} + 2 = \frac{n+2}{2}$. □

Lemma 3.2 *Let n be even. In P_{n+1}, it is possible for $e(0) = 1, 3, \cdots, M$, where $M = \frac{n+2}{2}$ if $n \equiv 0 \pmod 4$, and $M = \frac{n}{2}$ if $n \equiv 2 \pmod 4$, with $v(0) - v(1) = -1$ and with complementary labels for the end-vertices.*

Proof. Label the vertex v_1 by 1, and the vertices of P_{n+1} from v_2 to v_{n+1} alternately with 1's and 0's in this order. Obviously $v(0) - v(1) = -1$, and $e(0) = 1$ here.

First consider $n \equiv 0 \pmod 4$. Divide the vertices $v_2, \cdots, v_{n+1}$ into $\frac{n}{4}$ blocks of 4, each with vertex labels $1, 0, 1, 0$. Start with the block v_2, v_3, v_4, v_5 and change the vertex labels to $1, 1, 0, 0$ respectively. Then $e(0) = 3$. Continue with the rest of the blocks, and $e(0)$ increases by 2 each time until it finally reaches $\frac{n+2}{2}$.

Next consider $n \equiv 2 \pmod 4$. Divide the vertices $v_4, \cdots, v_{n+1}$ into $\frac{n-2}{4}$ blocks of 4. Perform exactly the same procedure as in the last case to these blocks of 4 to reach $e(0) = \frac{n}{2}$. □

Lemma 3.3 *Consider $W(a,b,c)$, where all of a, b, and c are even. Then $a+b+c+1 \notin FI(W(a,b,c))$.*

Proof. In order for $a+b+c+1$ to be in the friendly index set, either $e(0) = 1$ or $e(1) = 1$. Since a cycle must have an even number of 1-edges, the sole 0-edge or the sole 1-edge cannot be an edge of the cycle. If the sole 0-edge is a spoke, then the 3 corner vertices cannot all have the same label and the edges of the cycle cannot all be labeled 1. If the sole 1-edge is a spoke, then again the 3 corner vertices cannot all have the same label and the edges of the cycle cannot all be labeled 0. □

Lemma 3.4 *Consider $W(a,b,c)$, where all of a, b, and $c > 2$ and $\equiv 0 \pmod 4$. Then $1 \in FI(W(a,b,c))$.*

Proof. Start at any corner of the triangle, and label the vertices of the triangle using blocks of $1, 1, 0, 0$. Then each corner vertex will have the label 1 at the beginning of some 4-vertex block, and the triangle has equal numbers of 0-vertices and 1-vertices, and equal numbers of 0-edges and 1-edges. Label the center vertex 0. This vertex labeling is friendly with $v(0) - v(1) = 1$, and $e(0) - e(1) = -3$. Now change a corner vertex to 0. Then $v(0) - v(1) = -1$, and $e(0) - e(1) = -1$. □

Theorem 3.5 *Consider $W(a,b,c)$ where all of a, b and c are even and > 2. Then $FI(W(a,b,c)) = \{1, 3, \cdots, a+b+c-3, a+b+c-1, a+b+c+3\}$.*

Proof. By Theorem 1.1, 1.3, and Lemma 3.3, it suffices to show that these values are attainable.

Use the initial step in Lemma 3.1 to label the three sides of the triangle and label the center vertex 1. This vertex labeling is friendly, with $e(0) = 0$ and $e(1) = a+b+c+3$. Apply Lemma 3.1 to each of the three sides of the triangle to change $e(0)$ in increments to 2 up to the maximum values. Then for the last labeling, $e(0) \geq \frac{a+b+c}{2}$, $e(1) \leq \frac{a+b+c}{2} + 3$, and $e(1) - e(0) \leq 3$. So

$e(1)-e(0)$ changes from $(a+b+c+3)$ to this minimum value in decrements of 4. We may miss the number 1 if all of a, b and $c \equiv 0 \pmod 4$. However Lemma 3.4 takes care of this case.

Use the initial step in Lemma 3.2 to label two of the three sides of the triangle, the initial step in Lemma 3.1 (with complementary vertex labels) to label the third side, and label the center vertex 0. This vertex labeling is friendly, with $e(0)=3$, $e(1)=a+b+c$, and $e(1)-e(0)=a+b+c-3$. Apply Lemmas 3.1 and 3.2 to each of the three sides of the triangle to change $e(0)$ in increments to 2 up to the maximum values. Then for the last labeling, $e(0) \geq \frac{a+b+c}{2}$, $e(1) \leq \frac{a+b+c}{2}+3$, and $e(1)-e(0) \leq 3$. So $e(1)-e(0)$ changes from $(a+b+c-3)$ to this minimum value in decrements of 4. We may still miss the number 1. If all of a, b and $c \equiv 0 \pmod 4$, Lemma 3.4 takes care of it. If at least one of the three sides of the triangle is $\equiv 2 \pmod 4$, use this side as the third side referred to in the initial step in this paragraph. □

Example 7. Figure 7 shows that $\mathrm{FI}(W(2,2,2)) = \{1,3,5,9\}$. We note here that 7 is missing.

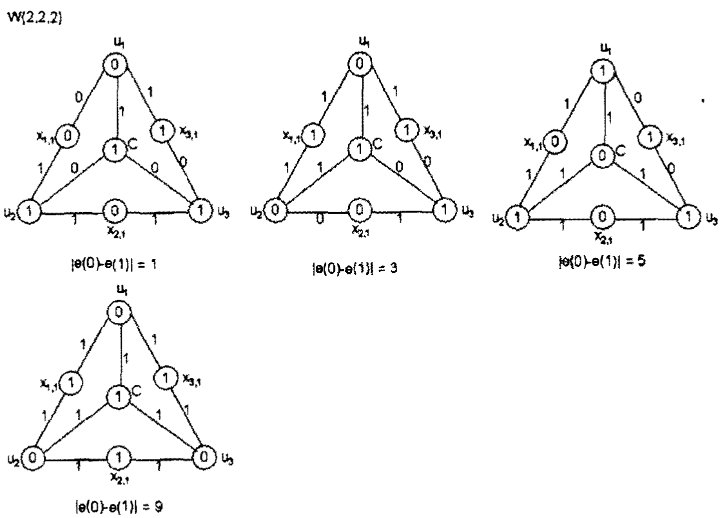


Figure 7: $\mathrm{FI}(W(2,2,2))$

Example 8. Figure 8 shows that $\mathrm{FI}(W(4,4,6)) = \{1,3,5,7,9,11,13,17\}$. We note here that 15 is missing.

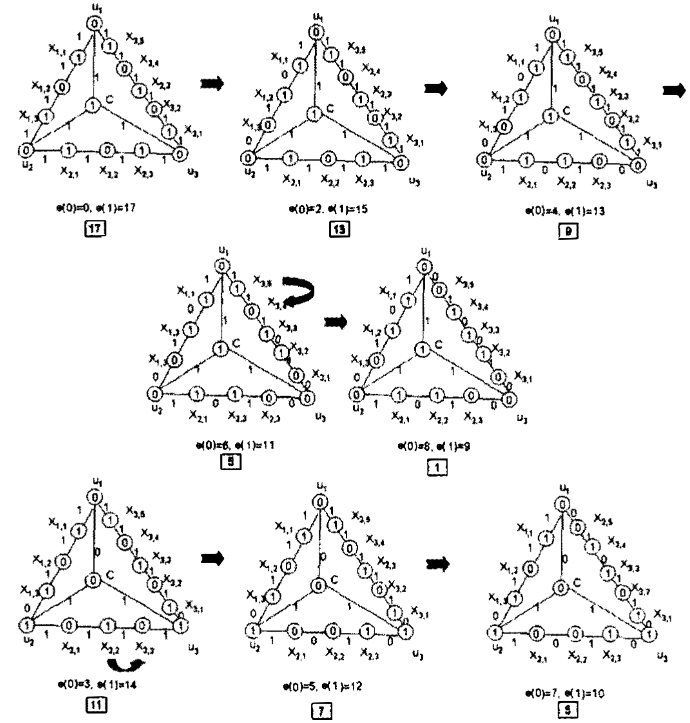


Figure 8: $\mathrm{FI}(W(4,4,6))$

4 Friendly index set of broken wheel $W(a,b,c)$, where exactly two of a,b,c are even

Theorem 4.1 *Consider $W(a,b,c)$ where a is odd, and b and c are even and >2. Then $FI(W(a,b,c)) = \{0,2,4,\cdots,a+b+c-3,a+b+c-1,a+b+c+1\}$.*

Proof. By Theorems 1.1 and 1.3, it suffices to show that these values are attainable.

Use the initial steps in Lemma 2.2 to label P_{a+1}, and Lemma 3.1 to label P_{b+1} and P_{c+1}. For the triangle, all three corner vertices have label 0, $v(0)-v(1)=1$, and all but one of the edges have induced label 1. Label the center vertex 1, making the vertex labeling friendly, and keeping $e(0)=1$ for the entire $W(a,b,c)$. Then $e(1)-e(0)=a+b+c+1$. Apply Lemmas 2.2 and 3.1 to the three sides of the triangle. The value of $e(0)$ can be changed in increments of 2 to at least $\frac{a-1}{2}+\frac{b}{2}+\frac{c}{2} = \frac{a+b+c-1}{2}$. Then $e(1) \leq \frac{a+b+c+7}{2}$, and so $e(1) - e(0)$ changes in decrements of 4 to a value that is ≤ 4.

Now use the initial steps in Lemma 2.1 to label P_{a+1}, Lemma 3.2 (in reverse order and with complementary vertex labels) to label P_{b+1}, and Lemma 3.1 to label P_{c+1}. For the triangle, exactly two of the three corner vertices have label 0, $v(0) - v(1) = 1$, and all but one of the edges have induced label 1. Label the center vertex 1, making the vertex labeling friendly, and making $e(0) = 2$ for the entire $W(a, b, c)$. Then $e(1) - e(0) = a + b + c - 1$. Apply Lemmas 2.1, 3.1, and 3.2 to the three sides of the triangle. The value of $e(0)$ can be changed in increments of 2 to at least $\frac{a-1}{2} + \frac{b}{2} + \frac{c}{2} + 1 = \frac{a+b+c+1}{2}$. Then $e(1) \leq \frac{a+b+c+5}{2}$, and so $e(1) - e(0)$ changes in decrements of 4 to a value that is ≤ 2.

If $a+b+c+1 \equiv 2 \pmod 4$, and so $a+b+c-1 \equiv 0 \pmod 4$, then $e(1)-e(0)$ in the previous two paragraphs actually changes in decrements of 4 to values that are ≤ 2 and 0 respectively, and the proof is complete. The only remaining case is when $a+b+c+1 \equiv 0 \pmod 4$, and so $e(1)-e(0)$ in the first algorithm may actually stop at 4, and 0 may not be reached using the algorithm. Going back to Lemmas 2.2 and 3.1, we see that this problem possibly exists only if $a \equiv 3 \pmod 4$, and both b and $c \equiv 0 \pmod 4$. If so, label the vertices of P_{a+1} by blocks of $1, 1, 0, 0$, label P_{b+1} (starting from the second vertex, because its first vertex is shared with P_{a+1} and has been labeled) by blocks of $1, 1, 0, 0$, and label P_{c+1} (starting from the second vertex and ending at the second last vertex, because its end-vertices are shared with P_{b+1} and P_{a+1} and have been labeled) by blocks of $1, 1, 0, 0$, with the last 0 missing. For the triangle, exactly two of the three corner vertices are labeled 0, $v(0) - v(1) = -1$, and $e(1) - e(0) = 1$. Label the center vertex by 0 to make $W(a, b, c)$ vertex friendly. Then exactly two of the three spokes have induced edge label 0. Thus $W(a, b, c)$ have the same number of 0-edges and 1-edges. This makes sure that 0 is also in the friendly index set. □

Example 9. Figure 9 shows that $\mathrm{FI}(W(5,4,6)) = \{0, 2, 4, 6, 8, 10, 12, 14, 16\}$.

5 Friendly index set of broken wheel $W(a, b, c)$, where exactly one of a, b, c is even

Theorem 5.1 *Consider $W(a, b, c)$ where a, b are odd, and $c > 2$ is even. Then $FI(W(a, b, c)) = \{1, 3, \cdots, a+b+c-3, a+b+c-1, a+b+c+1\}$.*

Proof. By Theorems 1.1 and 1.3, it suffices to show that these values are attainable.

Since $a + b + c$ is even, we can label the vertices of the triangle alternately by 0 and 1 so that all induced edge labels of the triangle are 1's.

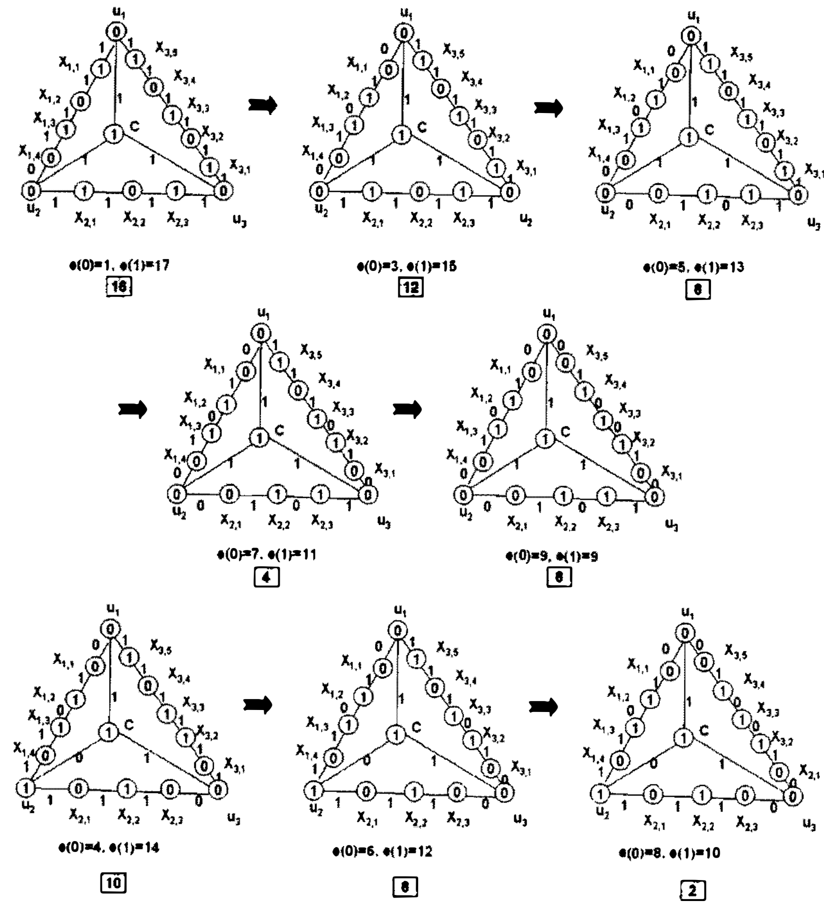


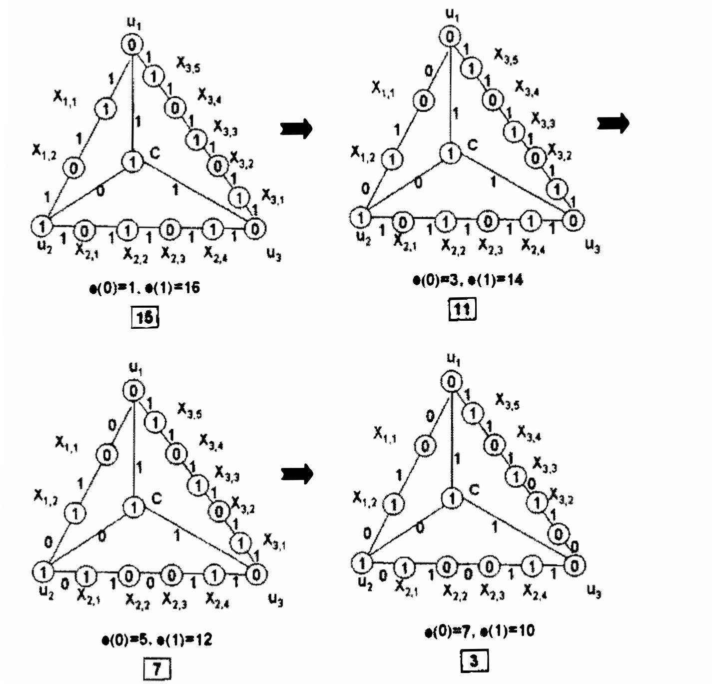
Figure 9: $\mathrm{FI}(W(5,4,6))$

Furthermore, by using Lemmas 2.1 and 3.1, we can do it in such a way that two of the corner vertices in the triangle have the label 0 whereas the remaining corner vertex of the triangle has the label 1.

Label the center vertex by 1. For $W(a, b, c)$, $e(1) = a+b+c+2$, $e(0) = 1$, and so $e(1) - e(0) = a + b + c + 1$. Using Lemmas 2.1 and 3.1, $e(0)$ can be changed in increments of 2 up to at least $\frac{a-1}{2} + \frac{b-1}{2} + \frac{c}{2} + 1 = \frac{a+b+c}{2}$. Then $e(1) \leq \frac{a+b+c}{2} + 3$, and so $e(1) - e(0)$ changes in decrements of 4 to a value that is ≤ 3.

Next, use the above initial labeling, except that the center vertex is labeled by 0. We have one additional 0-edge, resulting in an initial $e(1) - e(0) = a + b + c - 1$. Using exactly the same technique, we can change $e(1) - e(0)$ in decrements of 4 to a value that is ≤ 1. □

Example 10. Figure 10 shows displays $\mathrm{FI}(W(3, 5, 6)) = \{1, 3, 5, 7, 9, 11, 13, 15\}$.



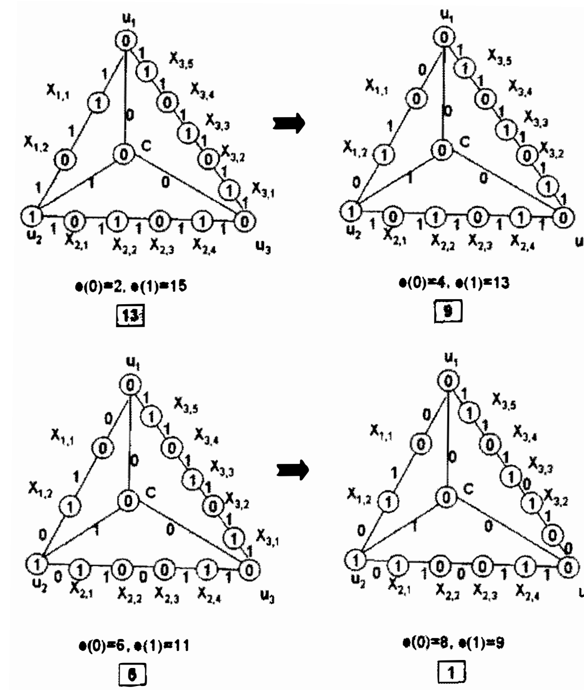


Figure 10: $\mathrm{FI}(W(3, 5, 6))$

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