# On Edge-Graceful and Edge-Magic Maximal Outerplanar Graphs

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## ABSTRACT

Let G be a (p,q)-graph in which the edges are labeled 1,2,3,...q. The vertex sum for a vertex v is the sum of the labels of the incident edges at v. If G can be labeled so that the vertex sums are distinct, mod p, then G is said to be edge-graceful. If the edges of G can be labeled 1,2,3,...q so that the vertex sums are constant, mod p, then G is said to be edge-magic. It is conjectured by Lee [9] that any connected simple (p,q)-graph with $q(q+1)\equiv p(p-1)/2 \pmod p$ vertices is edge-graceful. We show that the conjecture is true for maximal outerplanar graphs. We also completely determine the edge-magic maximal outerplanar graphs.

**1. Introduction**. All graphs in this paper are simple graphs with no loops or multiple edges. Graceful labelings were first introduced by Alex Rosa as a means of attacking the problem of cyclically decomposing the complete graph into other graphs. Since Rosa's original article, literally more than six hundreds of papers have been written on graph labelings [3]. A similar concept to graceful labelings of graphs, called edge-gracefulness, was introduced by S.P. Lo [21] in 1985.

A graph G=(V, E) with p vertices and q edges is said to be **edge-graceful** if there is a bijection $f : E \to \{1, 2,..., q\}$ such that the induced mapping $f^+: V \to Z_p$, given by $f^+(u) = \Sigma_v\{f(u,v): (u,v) \text{ in } E\} \pmod p$ is a bijection. The values $f^+(u)$ are called *vertex sums*. Figure 1 shows a (12,17)-graph with two different edge-gracefu labelings.

A necessary condition for edge-gracefulness is given by Lo [21]:

$$q(q+1)\equiv \pmod p$$

(1)

This latter condition may be more practically stated as $q(q+1) \equiv 0$ or $p/2 \pmod p$ depending on whether p is odd or even.

(2)

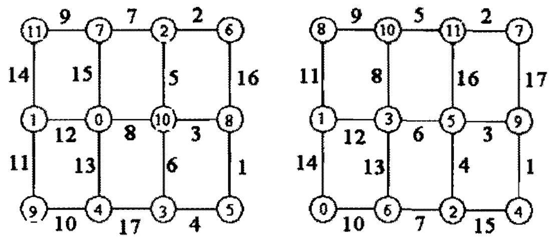


**Figure 1.**

Lee, Lee, Murthy [7] showed that if G is a (p,q)-graph with p≡ 2 (mod 4) then G is not edge graceful.
(3)

Lee had proposed the following tantalizing conjecture

**Conjecture** (Lee [9]): The Lo condition (1) is sufficient for a connected graph to be edge-graceful.

A sub-conjecture of this has also not yet been proved:

**Conjecture** (Lee [8 ]): All trees of odd-order are edge-graceful.

In order to work on these conjectures, another concept of edge-magic, similar to edge-gracefulness was introduced by Lee[14]. Let G be a (p,q) graph in which the edges are labeled 1,2,3,...q so that the vertex sums are constant, mod p. Then G is said to be **edge-magic.** The concept of edge-magic graphs was introduced by the first author, Seah and Tan [16]. A necessary condition for a (p,q)-graph to be edge-magic is $q(q+1)\equiv 0 \pmod p$. However, this condition is not sufficient. There are infinitely many connected graphs such as trees and cycles satisfying this condition that are not edge-magic.

The cartesian product of two paths is frequently called the **grid** graph. The cartesian product of two cycles is called the **torus** graph. It was shown in [25] that the torus graph $C_m \times C_n$ is edge-magic for all m,n >2.

Lee, Pigg and Cox [11] showed that $C_n \times K_2$ is edge-magic if and only if n is odd and ≥3.

Karl Schaffer and Sin-Min Lee [25] have shown that if G and H are both odd-order, regular, edge-graceful graphs, where G is d-regular with m vertices, H is k-regular with n vertices, and GCD(d,n) = GCD (k,m) = 1, then G x H is edge-graceful. In particular, they showed that the torus graph $C_{2i+1} \times C_{2j+1}$ is edge-graceful.

In 1993 Lee conjectured that every connected simple cubic graph G with p≡ 2 (mod 4) is edge-magic. Lee, Pigg and Cox [11] showed that the conjecture is true for prisms and other cubic graphs. In [20], it is shown the conjecture is not true for p=10.

A planar graph is called an outerplanar graph if in a plane embedding its vertices can be placed on the boundary of a face. This face is usually called the outer face. The edges on the boundary of an outerplanar graph are called outer edges and other edges are called inner edges or chords.



Chartrand and Harary [2] showed that a graph is outerplanar if and only if it does not contains a $K_4$ or $K_{2,3}$ minor. An outerplanar graph is said to be maximal if it has the maximum possible number of edges for the given number of vertices. A maximal outerplanar graph can be viewed as a triangulation of a convex polygon (Figure 2)

We record some facts about maximal outerplanar graphs. The reader is referred to [4] for details of the proof of the following lemma.

**Lemma**. Let G be a maximal outerplanar graph with n vertices, n≥3. Then:
(i) there are 2n-3 edges, of which there are n-3 chords;
(ii) there are n-2 inner faces. Each inner face is a triangle;
(iii) there are at least two vertices with degree 2;

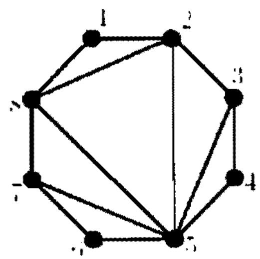


Figure 2

The problem of finding an edge-graceful labeling or an edge-magic labeling of a graph is very difficult. Several classes of graphs have been shown to be edge-graceful [1,5,6,7,8,10,12,13,14,15,17,18,19,21,23,24]. For more conjectures and open problems on edge-magic graphs the reader is referred to [11,16,17,20,27,28]. The reader should also see the survey article of Gallian [3] for various labeling problems.

## 2. Edge-magic Maximal Outerplanar Graphs.

**Theorem 1**. A maximal outerplanar graph with p vertices is edge-magic only if p=6.

**Proof.** A maximal outerplanar graph with p vertices is edge-magic only if it satisfies

$$q(q+1)\equiv 0 \pmod p$$

$\Rightarrow (2p-3)(2p-2)\equiv 0 \pmod p$

$\Rightarrow (4p-6)(p-1)\equiv 0 \pmod p$

$\Rightarrow 4p-6\equiv 0 \pmod p$

$\Rightarrow 6\equiv 0 \pmod p$

Thus p is 6.

**Theorem 2**. All maximal outerplanar graph with 6 vertices are edge-magic.

**Proof**. Up to isomorphism there are three maximal outerplanar graphs of order 6.

(Figure 3)

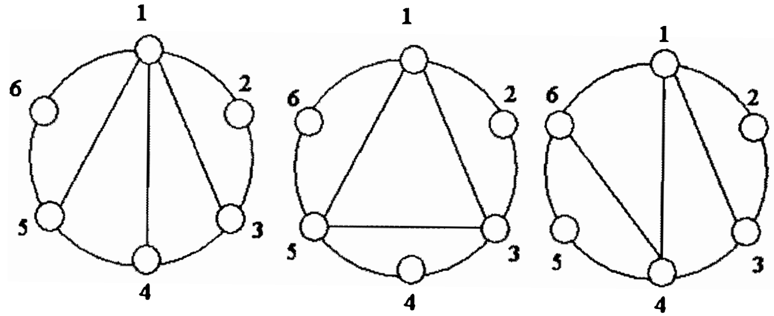


**Figure 3.**

Figure 4 shows that they are edge-magic.

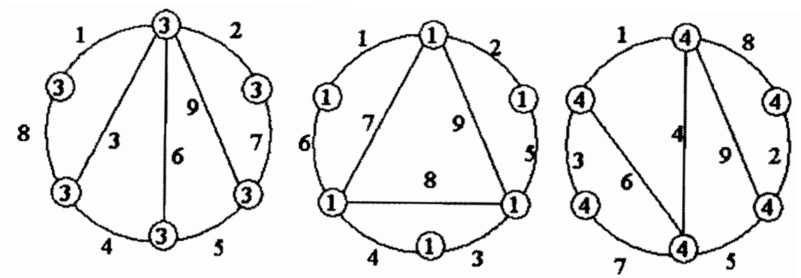


**Figure 4.**

## 3. Edge-graceful Maximal Outerplanar Graphs.

Finding an edge-graceful labeling of a graph is related to solving a system of linear Diophantine equations. In general it is difficult to find an edge-graceful labeling of a graph.

**Theorem 3**. A maximal outerplanar graph with p vertices is edge-graceful only if p = 4 or 12.

**Proof.** A maximal outerplanar graph with p vertices has 2p-3 edges. It is edge-graceful only if it satisfies Lo's condition



$$q(q+1)\equiv p(p-1)/2 \pmod p$$

$\Rightarrow (2p-3)(2p-2)\equiv p(p-1)/2 \pmod p$

$\Rightarrow (4p-6)(p-1)\equiv p(p-1)/2 \pmod p$

$\Rightarrow 4p-6\equiv p/2 \pmod p$

Thus p is even, say p=2k. We have $7k\equiv 6 \pmod{2k}$, i.e. $k\equiv 6 \pmod{2k}$.

From which we have k=2 or k=6. Thus p = 4 or 12.

**Theorem 4.** The maximal outerplanar graph with 4 vertices is edge-graceful.

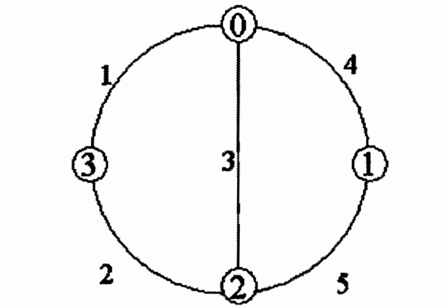


**Figure 5.**

**Proof.** Figure 5 shows that it is edge-magic.

In order to show that all maximal outerplanar graphs with 12 vertices are edge-graceful, we constructed the maximal outerplanar graphs by computer, beginning with the unique maximal outerplanar graph on 5 vertices. Write MOP(p) for the number of mutually non-isomorphic maximal outerplanar graphs on p vertices. Then MOP(4) = MOP(5) = 1. Let G be a maximal outerplanar graph on p vertices. Then G has a vertex u of degree 2. Let v, w be the vertices adjacent to u. Since G is a near triangulation, vw is an edge of G. It follows that G-u is a maximal outerplanar graph on p-1 vertices. Therefore we can construct all maximal outerplanar graphs on p vertices from those on p-1 vertices, by the following algorithm,

for each maximal outerplanar graph G on p-1 vertices do begin
    let $v_0, v_1, \ldots, v_{p-2}$ be the outer cycle of G
    for k=0 to p-2 do begin
        let $G' = G + v_p v_k + v_p v_{k+1}$ (subscript addition is mod p-1)
        G' is now a maximal outerplanar graph on p vertices
        write G' to a file

end

end

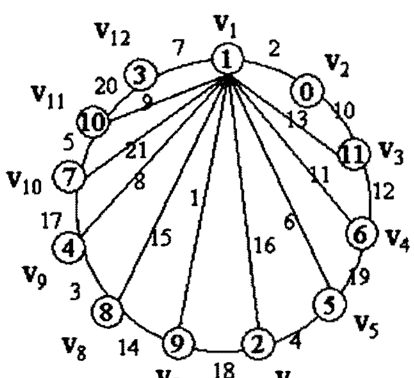
This loop will construct p-1 maximal outerplanar graphs G' on p vertices, for each maximal outerplanar graph G on p-1 vertices. Every maximal outerplanar graph on p vertices will be constructed in this way.

However, many generated graphs are isomorphic. In order to remove duplicates, the output file of this algorithm is input to the graph isomorphism program in Groups&Graphs (**bkocay.cs.umanitoba.ca/G&G/G&G. html**). The result is a file of all mutually non-isomorphic maximal outerplanar graphs on p vertices. The process is then repeated, inputting this file to the same algorithm, until all maximal outerplanar graphs on p=12 vertices have been constructed. The number of graphs found is as shown.

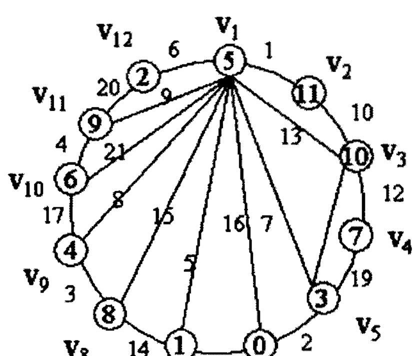
| p | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| MOP(p) | 1 | 1 | 3 | 4 | 12 | 27 | 82 | 228 | 733 | 2282 | 7528 |

**Theorem 5.** The maximal outerplanar graphs on 12 vertices are edge-graceful.

This was done by a computer program. All the 733 maximal outerplanar graphs were input to our program which exhaustively searches for an edge-graceful labeling. All graphs were found to have edge-graceful labelings. Some of them are shown below.

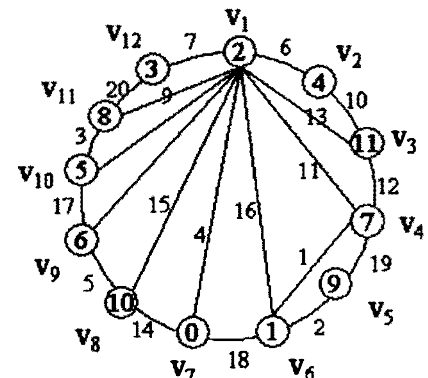


$M_1$

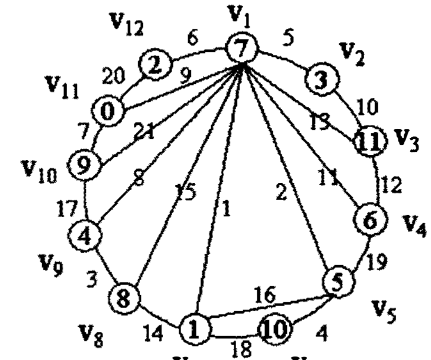


$M_2$

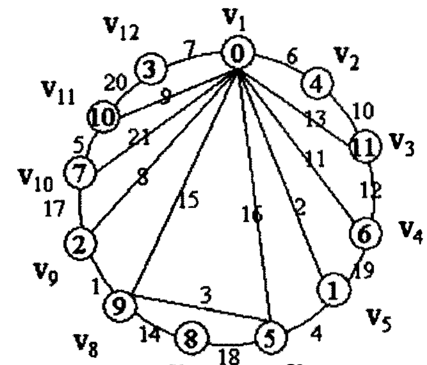




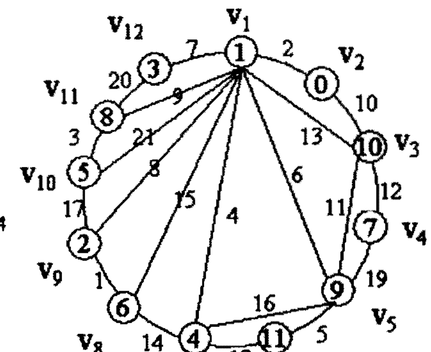
$M_3$



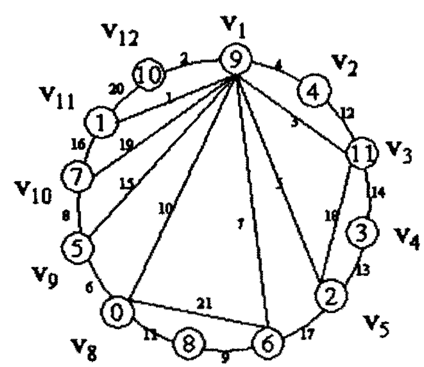
$M_4$



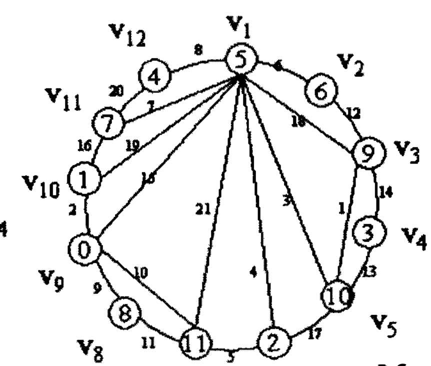
$M_5$



$M_6$



$M_7$



$M_8$

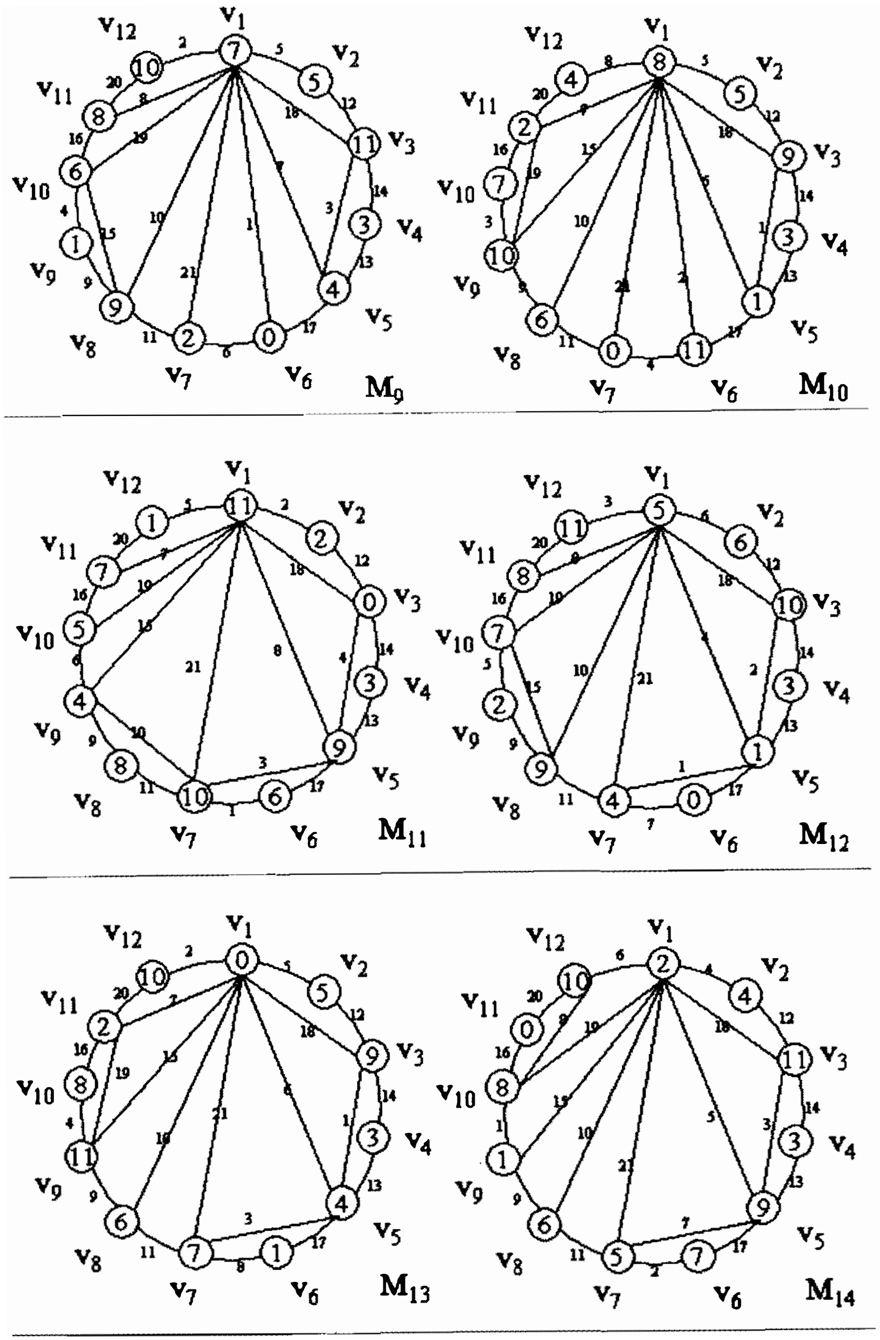



**Figure 6.**

All the 733 maximal outerplanar graphs and their edge-graceful labelings are post in the web page of Sin-Min Lee. The reader can see directly from: www.cs.sjsu.edu/faculty/lee

**Remark.** To minimize overhead time, we use the *iterative* version of Sedgewick's heap method algorithm to generate permutations. To test every permutation sequentially is unsound. In general, find an edge-graceful labeling for one graph will take 3 days. We observe the order in which the edges are input is matter. If an edge-graceful label is not found to one graph after a few thousand permutations, we will randomize the input and try again. In this way we can obtain all the labelings of 733 graphs in less than one day.

# References

[1] S. Cabannis, J. Mitchem and R. Low, On edge-graceful regular graphs and trees., *Ars Combin.* **34,** 129-142 ,1992.

[2] G. Chartrand and F. Harary, Planar permutation graphs, *Ann. Inst. Henri Poincare* B. 433-438, 3, 1967..

[3] J.A. Gallian, A dynamic survey of graph labeling, *The Electronic J. of Combin.* (2003), # DS6, 1-144.

[4] F. Harary. Graph Theory. Addison-Wesley Publishing Company, 1971.

[5] Jonathan Keene and Andrew Simoson, Balanced strands for asymmetric, edge-graceful spiders, *Ars Combinatoria* **42**,49-64,1996

[6] Q. Kuan, Sin-Min Lee, J. Mitchem, and A.K. Wang, On edge-graceful unicyclic graphs, *Congressus Numerantium* **61**, 65-74, 1988.

[7]Li Min Lee, Sin Min Lee, and G. Murty, On edge-graceful labellings of complete graphs - solutions of Lo's conjecture, *Congressum Numerantum* **62**, 225-233, 1988.

[8] Sin-Min Lee, A conjecture on edge-graceful trees, *Scientia*, Ser. A, vol.3, 45-57, 1989.

[9] Sin-Min Lee, New Directions in the Theory of Edge-Graceful Graphs, *Proceedings of the 6th Caribbean Conference on Combinatorics & Computing*, pp 216-231, 1991.

[10] Sin-Min Lee, Peining Ma, Linda Valdes, and Siu-Ming Tong, On the edge-graceful grids, Congressus Numerantium **154**,61-77, 2002.

[11] Sin-Min Lee, W.M. Pigg and T.J. Cox, On edge-magic cubic graphs conjecture, *Congressus Numerantium*, **105** ,214-222, 1994.

[12] Sin-Min Lee and Eric Seah, Edge-graceful labellings of regular complete k-partite graphs, *Congressum Numerantum* **75**, 41-50, 1990.

[13] Sin-Min Lee and Eric Seah, On edge-gracefulnes of the composition of step graphs with null graphs, *Combinatorics, Algorithms, and Applications in Society for Industrial and Applied Mathematics,*, 326-330, 1991.

[14] Sin-Min Lee and Eric Seah, On the edge-graceful (n,km)-multigraphs conjecture, *Journal of Combinatorial Mathematics and Combinatorial Computing*, Vol. 9, 141-147, 1991.

[15] Sin-Min Lee, E. Seah and S.P. Lo , On edge-graceful 2-regular graphs, *The Journal of Combinatoric Mathematics and Combinatoric Computing*, **12**, 109-117,1992.

[16] Sin-Min Lee, Eric Seah and S.K. Tan, On edge-magic graphs, *Congressus Numerantium,* **86,** 179-191, 1992.

[17] Sin-Min Lee, E. Seah, Siu-Ming Tong, On the edge-magic and edge-graceful total graphs conjecture, *Congressus Numerantium* **141**, 37-48 ,1999

[18] Sin-Min Lee, E. Seah and P.C. Wang , On edge-gracefulness of the kth power graphs, *Bulletin of the Institute of Math*, Academia Sinica 18, No. 1, 1-11, 1990.

[19] Sin-Min Lee, Ling.Wang and K. Nowak, On super edge-graceful trees, manuscript.

[20] Sin-Min Lee, Ling.Wang and Yihui Wen, On The Edge-magic Cubic Graphs and Multigraph, *Congressus Numerantium* **165** ,145 – 160, 2003.

[21] S.P. Lo, On edge-graceful labelings of graphs, *Congressus Numerantium,* **50**, 231-241, 1985.

[22] J. Mitchem and A. Simoson, On edge-graceful and super-edge-graceful graphs. *Ars Combin.* **37**, 97-111,1994.

[23] Jin Peng. and W. Li, Edge-gracefulness of $C_m x C_n$, in *Proceedings of the Sixth Conference of Operations Research Society of China*, (Hong Kong: Global-Link Publishing Company), Changsha, October 10-15, p942-948, 2000.

[24] A. Riskin and S. Wilson, Edge graceful labellings of disjoint unions of cycles. *Bulletin of the Institute of Combinatorics and its Applications* **22**: 53-58,1998.

[25] Karl Schaffer and Sin Min Lee, Edge-graceful and edge-magic labellings of Cartesian products of graphs, *Congressus Numerantium* **141**, 119-134, 1999.



[26] W.C. Shiu, P.C.B. Lam and Sin-Min Lee, Edge-magicness of the composition of a cycle with a null graph, *Congressus Numerantium,* **132** ,9-18, 1998.

[27] W.C. Shiu, F.C.B. Lam and Sin-Min Lee, Edge - magic index sets of (p, p-1)-graphs, *Electronic Notes in Discrete Mathematics*, Vol. 11 , 2002.

[28] W.C. Shiu and Sin-Min Lee, Some edge-magic cubic graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*, **40** , 115-127, 2002.

[29] W.C. Shiu, Sin-Min Lee and K. Schaffer, Some k-fold edge-graceful labellings of (p,p-1)-graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing* **38**, 81-95, 2001.

[30] S. Wilson and A. Riskin, Edge-graceful labellings of odd cycles and their products,, *Bulletin of the ICA*, **24**, 57-64, 1998.