ON THE INTEGER–MAGIC SPECTRA OF MAXIMAL PLANAR AND MAXIMAL OUTERPLANAR GRAPHS

SIN–MIN LEE, YONG–SONG HO, AND RICHARD M. LOW

ABSTRACT. For $k \geq 2$, a graph $G = (V, E)$ is called Z_k–**magic** if there exists a labeling $f : E(G) \to Z_k^*$ such that the induced vertex set labeling $f^+ : V(G) \to Z_k$, defined by $f^+(v) = \Sigma f(u, v)$ where $(u, v) \in E(G)$, is a constant map. In this paper, we investigate $\{k : G \text{ is } Z_k\text{–magic}, k \geq 2\}$ for maximal planar and maximal outerplanar graphs G.

1. INTRODUCTION

Let G be a connected graph without multiple edges or loops. For any abelian group A (written additively), let $A^* = A - \{0\}$. A function $f : E(G) \to A^*$ is called a *labeling* of G. Any such labeling induces a map $f^+ : V(G) \to A$, defined by $f^+(v) = \Sigma f(u, v)$ where $(u, v) \in E(G)$. If there exists a labeling f which induces a constant label c on $V(G)$, we say that f is an *A–magic labeling* and that G is an *A–magic* graph with *index* c. The set $\{k : G \text{ is } Z_k\text{–magic}, k \geq 2\}$ is called the *integer–magic spectrum* of a graph G. Although this paper does not directly address Z–magic graphs, these graphs can be viewed as Z_1–magic graphs.

Z–magic graphs were considered by Stanley [17,18], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous diophantine equations. Doob [2,3,4] and others [8,11,13] have studied A–magic graphs and Z_k–magic graphs were investigated in [5,9,10,12].

Within the mathematical literature, various definitions of magic graphs have been introduced. The original concept of an A–magic graph is due to J. Sedlacek [14,15], who defined it to be a graph with real–valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices. Previously, Kotzig and Rosa [6] had introduced yet another definition of a magic graph. Over the years, there has been

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great research interest in graph labeling problems. The interested reader is directed to Wallis' [19] recent monograph on magic graphs.

2. Maximal planar graphs

Informally, a *planar* graph is a simple graph which can be drawn in the plane without the crossing of edges. A planar graph is *triangulated* if and only if all its faces have three corners. We say that a graph is a *maximal planar* graph if it has the property that any further addition of edges results in a nonplanar graph. Clearly, a planar graph of order 3 or greater is maximal if and only if it is triangulated. Thus, a maximal planar graph with n vertices, $n \geq 3$, has $3n - 6$ edges.

Observation: Since $K_2 + P_2$ ($\cong K_4$) is a regular graph, its integer–magic spectrum is $N - \{1\}$.

Theorem 1. *The integer–magic spectrum of $K_2 + P_{2k}$ is $N - \{1, 2\}$, for all $k \geq 2$.*

Proof. Clearly, $K_2 + P_{2k}$ ($k \geq 2$) is not Z_2–magic. The labeling in Figure 1 gives a Z_n–magic labeling for all $n \geq 3$. Note that every vertex is labeled $2k + 3$.

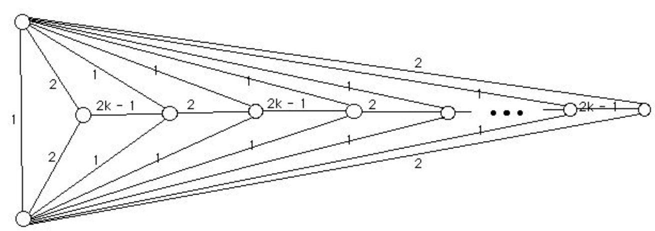

Figure 1.

□

Theorem 2. *The integer–magic spectrum of $K_2 + P_3$ is $N - \{1, 2\}$.*

Proof. Clearly, $K_2 + P_3$ is not Z_2–magic. Figure 2 gives Z_k–magic labelings, for all $k \geq 3$.

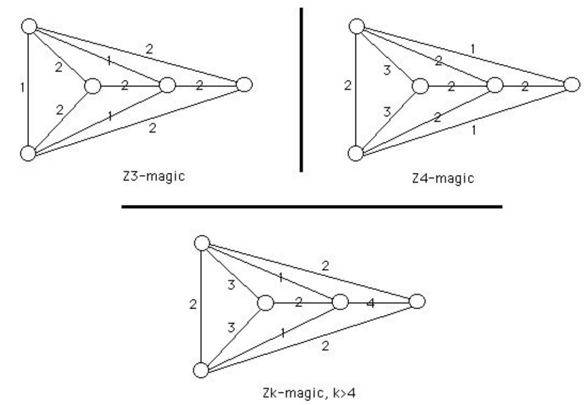

Figure 2.

□

Theorem 3. *The integer–magic spectrum of $K_2 + P_5$ is $N - \{1, 2\}$.*

Proof. Clearly, $K_2 + P_5$ is not Z_2–magic. Figure 3 illustrates Z_k–magic labelings, $k \geq 3$, for $K_2 + P_5$.

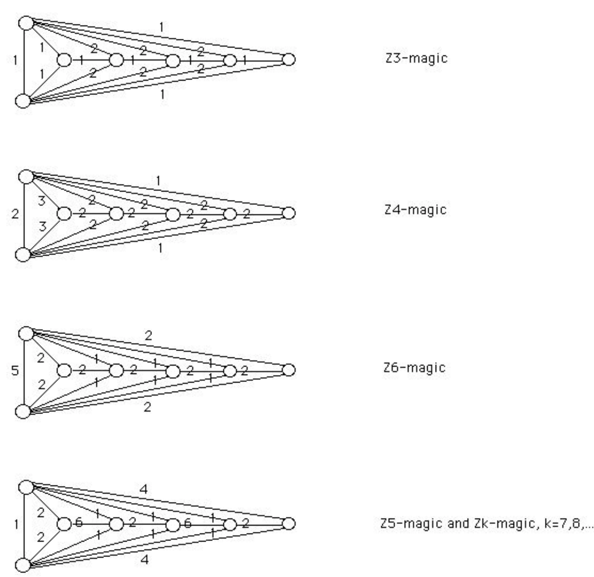

Figure 3.

□

3. Maximal outerplanar graphs

A planar graph is *outerplanar* if in a plane embedding, its vertices can be placed on the boundary of a face. This face is usually called the *outer face.* The edges on the boundary of an outerplanar graph are called *outer edges* and the other edges are called *inner edges* or *chords*. If we consider an outerplanar graph G with no loops or faces bounded by two edges, it may be possible to add a new edge to the presentation of G so that these properties are preserved. When no such adjunction can be made, G is a *maximal outerplanar* graph. A maximal outerplanar graph can be viewed as a triangulation of a convex polygon.

There are various characterizations of maximal outerplanar graphs. Chartrand and Harary [1] showed that a graph is outerplanar if and only if it does not contain a K_4 or $K_{2,3}$ minor. Kumar and Madhavan [7] gave a characterization of maximal outerplanar graphs, in the context of planar chordal graphs.

The reader should note the following observations:

Observations: Let G be a maximal outerplanar graph with n vertices, $n \geq 3$. Then, we have the following:
1. G has $2n-3$ edges, of which $n-3$ of them are chords.
2. G has $n-2$ inner faces. Each inner face is triangular.
3. G has at least two vertices of degree 2.
4. The connectivity of G is $\kappa(G) = 2$.

The next two results give us some information about the integer–magic spectrum of maximal outerplanar graphs.

Theorem 4. *For every maximal outerplanar graph G, there exists a Z_{2k}–magic labeling having index c, where $c \in 2N$, $2k \nmid \frac{c}{2}$, and $2k \nmid (\frac{c}{2}+k)$.*

Proof. We prove this by induction on $|V(G)|$. Note that if $|V(G)| = 3$, then $G \cong C_3$. Here, we label all edges of G with $\frac{c}{2}$. Thus, G has a Z_{2k}–magic labeling with index c, where $c \in 2N$, $2k \nmid \frac{c}{2}$, and $2k \nmid (\frac{c}{2}+k)$.

Now, assume that all maximal outerplanar graphs of order $n-1$ are Z_{2k}–magic with index c, where $c \in 2N$, $2k \nmid \frac{c}{2}$, and $2k \nmid (\frac{c}{2}+k)$. Let G be a maximal outerplanar graph, where $|V(G)| = n$ and v_s is a vertex of degree 2. Clearly, $\widehat{G} = G - \{v_s\}$ is a maximal outerplanar graph of order $n-1$. By the induction hypothesis, there exists a Z_{2k}–magic labeling $\widehat{g} : E(\widehat{G}) \to Z_{2k}^*$ with index c, where $c \in 2N$, $2k \nmid \frac{c}{2}$, and $2k \nmid (\frac{c}{2}+k)$. We define a labeling $g : E(G) \to Z_{2k}^*$ in the following manner:

CASE 1. $\widehat{g}((v_{s-1}, v_{s+1})) \neq \frac{c}{2}$, (mod $2k$).
Define $g((v_s, v_{s-1})) = g((v_s, v_{s+1})) = \frac{c}{2}$ and $g((v_{s-1}, v_{s+1})) = \widehat{g}((v_{s-1}, v_{s+1})) - \frac{c}{2}$. All other edges are labeled as in $\widehat{g}$. Note that $g : E(G) \to Z_{2k}^*$ does not label any edge with 0 and that every vertex has an induced labeling of c. Thus, G is Z_{2k}–magic.

CASE 2. $\widehat{g}((v_{s-1}, v_{s+1})) = \frac{c}{2}$, (mod $2k$).
Define $g((v_s, v_{s-1})) = g((v_s, v_{s+1})) = \frac{c}{2} + k$ and $g((v_{s-1}, v_{s+1})) = \widehat{g}((v_{s-1}, v_{s+1})) - \frac{c}{2} - k$. All other edges are labeled as in $\widehat{g}$. Note that $\frac{c}{2} + k \neq 0$, (mod $2k$) and $\widehat{g}((v_{s-1}, v_{s+1})) - \frac{c}{2} - k = -k \neq 0$, (mod $2k$). Furthermore, every vertex has an induced labeling of c. Thus, G is Z_{2k}–magic. □

Corollary 1. *Every maximal outerplanar graph is Z_{2k}–magic, for all $k \geq 2$.*

Proof. Let $c = 2$. If $k \geq 2$, then all of the hypothesis of Theorem 4 are satisfied and the result immediately follows. □

In [16], the number of non–isomorphic maximal outerplanar graphs with p (≥ 3) vertices is described by the following sequence: 1, 1, 1, 3, 4, 12, 27, 82, 228, 733, 2282, 7528,... We now establish the integer–magic spectrum of the maximal outerplanar graphs of small order.

Theorem 5. *The integer–magic spectrum of the maximal outerplanar graph of order 4 is $2N - \{2\}$.*

Proof. First, note that the maximal outerplanar graph of order 4 is isomorphic to $K_1 + P_3$. Suppose that $K_1 + P_3$ is Z_k–magic and has a Z_k–magic labeling as in Figure 4. Then, $b + (a + b) + (b + c) = a + b + c$. This implies that $2b = 0$. Clearly, $K_1 + P_3$ is not Z_2–magic. This, along with Corollary 1, implies that the integer–magic spectrum of $K_1 + P_3$ is $2N - \{2\}$.

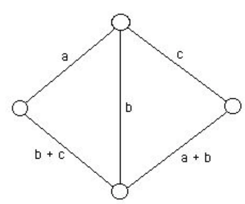

Figure 4.

□

Theorem 6. *The integer–magic spectrum of the maximal outerplanar graph of order 5 is $N - \{1, 2\}$.*

Proof. Note that the maximal outerplanar graph of order 5 is isomorphic to $K_1 + P_4$. Clearly, $K_1 + P_4$ is not Z_2–magic. Figure 5 shows that $K_1 + P_4$ is Z_k–magic, for $k = 3, 4, 5, 6$. The last labeling gives a Z_k–magic labeling, for all $k \geq 7$.

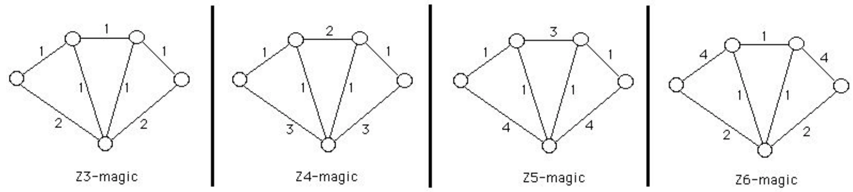

Figure 5.

□

There are three non–isomorphic maximal outerplanar graphs of order 6. (See Figure 6.) We now determine the integer–magic spectrum for these graphs.

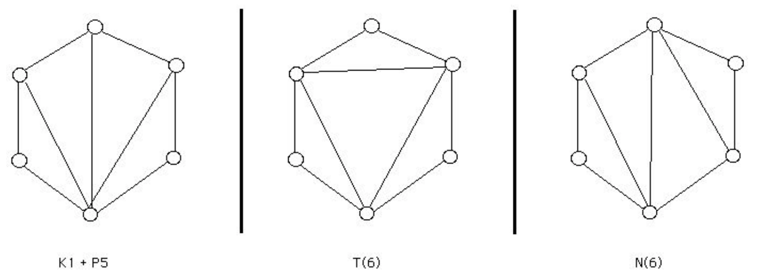

Figure 6.

Theorem 7. *The integer–magic spectrum of* $K_1 + P_5$ *is* $N - \{1, 2, 3\}$.

Proof. Clearly, $K_1 + P_5$ is not Z_2–magic. A straight–forward indirect proof can be used to show that $K_1 + P_5$ is not Z_3–magic. Figure 7 illustrates a Z_{2k+1}–magic labeling, for $k \geq 2$.

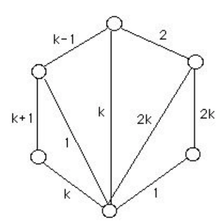

Figure 7. A Z_{2k+1}–magic labeling, $k \geq 2$.

This, along with Corollary 1, proves the theorem.

□

The integer-magic spectra for the two remaining cases are described by the following theorems.

Theorem 8. *The integer–magic spectrum of $T(6)$ is $N - \{1, 3\}$.*

Proof. In [13], Low and Lee showed that every eulerian graph is Z_{2k}–magic, for all $k \geq 1$. Using an indirect proof, it is straight–forward to show that $T(6)$ is not Z_3–magic. For brevity, this detail has been omitted. Figure 8 shows a Z_{2k+1}–magic labeling, $k \geq 2$, for $T(6)$:

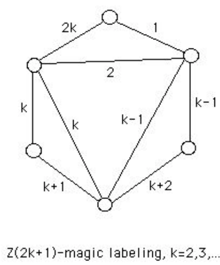

Figure 8.

Thus, the integer–magic spectrum of $T(6)$ is $N - \{1, 3\}$. □

Theorem 9. *The integer–magic spectrum of $N(6)$ is $N - \{1, 2\}$.*

Proof. Clearly, $N(6)$ is not Z_2–magic. Figure 9 illustrates Z_k–magic labelings, $k \geq 3$, for $N(6)$:

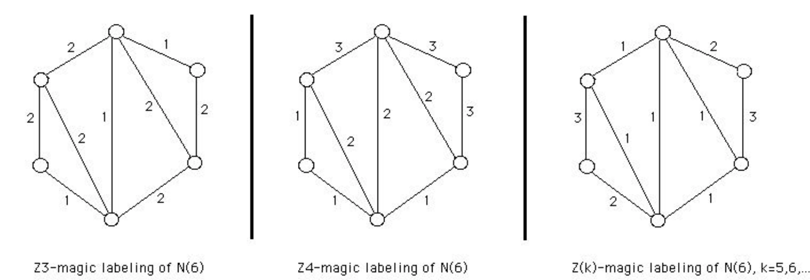

Figure 9.

Thus, the integer–magic spectrum of $N(6)$ is $N - \{1, 2\}$. □

4. An open problem

Open Problem 1. *For $n \geq 7$, determine the integer–magic spectra for the non–isomorphic, maximal outerplanar graphs of order n.*

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