On the edge-magic indices of (v, v+1)-graphs

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Dedicated to Prof. M.K. Siu on the occasion of his 60th birthday

Abstract Let G be a (v, q)-graph. If the edges can be labeled 1, 2, 3, ... ,q so that the vertex sums are constant (mod v), then G is said to be edge-magic. A necessary condition of edge-magicness is $v \mid q(q+1)$. In [9], it was shown that for any graph G there is an integer k such that the k-fold graph G[k] is edge-magic. The least such integer k is called the edge-magic index of G. We characterize k for some (v, v + 1)-graphs. If $v = p^n$, where p is a prime, then any $(p^n, p^n + 1)$-graph has $p^n - 1$ as its edge-magic index. We also show here that there are infinitely many (v, v + 1)-graphs with edge-magic index less than v – 1.

1. Introduction. In this paper we consider graphs, possibly pseudographs and disconnected graphs without isolated vertices.

A graph G = (V, E) with v vertices and q edges is said to be **magic** if there is a bijection $h : E \to \{1, 2,..., q\}$ such that the induced mapping $h^+: V \to Z$, given by $h^+(u) = \Sigma\{h(u, w) : (u, w) \text{ in } E\}$ is a constant. It is said to be **edge-magic** if

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there is a bijection $h : E \to \{1, 2, \dots, q\}$ such that the induced mapping $h^+: V \to Z_v$, given by $h^+(u) = \Sigma\{h(u, w) : (u, w) \text{ in } E\} \pmod v$ is a constant.

The concept of edge-magic graphs was introduced by the first author, Seah and Tan [9]. A necessary condition for a (v, q)-graph to be edge-magic is $q(q + 1) \equiv 0 \pmod v$. However, there are infinitely many connected graphs such as trees and cycles that satisfy the necessary condition but are not edge-magic.

The following figure shows three connected (6, 9)-graphs that are edge-magic. (Figure 1)

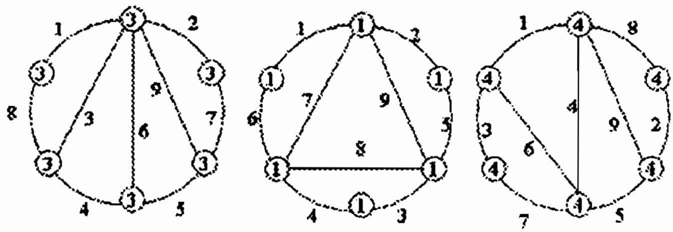


Figure 1.

It was shown in [15] that the torus graph $C_m \times C_n$ is edge-magic for all m, n > 2.

In this paper we introduce a construction of edge-magic graphs by splitting edges. In [22], we introduced a general construction of multigraphs from a given simple graph. Let $N = \{1, 2, 3, \dots\}$ be the set of natural numbers. Given a pair (G, f) where G = (V, E) is a simple graph and $f: E(G) \to N$, we can construct a graph SPE(G, f) as follows: To each edge e of E(G), if f(e) = k, we associate a set of parallel edges $P(e) = \{e_1, e_2, \dots, e_k\}$. We see that V(SPE(G,f)) = V(G) and $E(SPE(G, f)) = \cup\{P(e): e \in E(G)\}$.

We call the graph SPE(G, f) the **edge-splitting extension graph** of (G, f).

We illustrate here with an example:

Example 1. Let $G = C_4$ and $f : E(C_4) \to N$ be as follows:

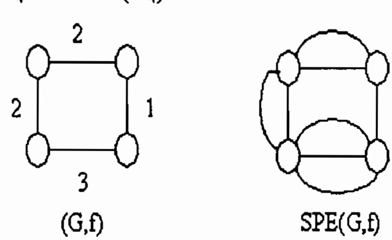


Figure 2.

If $f : E(G) \to N$ is a constant mapping, i.e., $f(e) = k$ for all edges e, we will denote SPE(G, f) simply by G[k]. It was shown in [9] that for any graph G, there exists a k such that G[k] is edge-magic. Denote by IM(G) the set $\{k : G[k]$ is edge-magic$\}$. The smallest k in IM(G) is called the **edge-magic index** of G.

Example 2. The following pseudograph of order 2 and size 3 has edge-magic index 1. Thus there exists a graph that is magic as well as edge-magic.

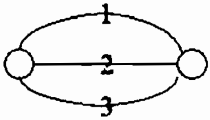


Figure 3.

The edge-magic indices of certain classes of graphs were considered in [16, 18, 19]. In this paper we determine the edge-magic indices of some $(v, v + 1)$-graphs. Since vertex labels are defined using addition modulo v, the integers 0 and v are the same. They will be used interchangeably in this paper.

2. Edge-magic indices of (v, v + 1)-graphs: Lower and upper bounds.

In this section we develop lower and upper bounds for the edge-magic index of a $(v, v + 1)$-graph G.

Theorem 1. If G is a $(v, v + 1)$-graph then its edge-magic index $\geq \min \{k : v \mid k(k + 1)\}$.

Proof. If m is the edge-magic index of G, then G[m] is edge-magic. We observe that $q = (v + 1)m$. By the necessary condition of edge-magicness referred to above, we have $(v + 1)m[(v + 1)m + 1] \equiv 0 \pmod{v}$, giving $m^2 + m \equiv 0 \pmod{v}$. Thus $v \mid m(m + 1)$.

Let S be a set. We use $S \times n$ to denote the multi-set of n copies of S. Note that S may be a multi-set itself. From now on the term "set" means "multi-set" and set operations are multi-set operations.

Let S be a set of qk elements. A partition $\wp$ of S is called a (q, k)-partition of S if each of the q classes in the partition contains k elements.

Notation. We let $[0] = \phi$, and for any integer $r > 0$, $[r] = \{1, 2, \ldots, r\}$.

Suppose A is a set consisting of r integers, and its sum is s modulo v, we will say that A is an (s; r)-set or simply an s-set.

A mapping f is called a k-fold edge-magic labeling of a (v, q)-graph G if there is a (q, k)-partition $\wp$ of [qk] such that $f : E(G) \to \wp$ is a bijection and the induced mapping $f^+ : V(G) \to Z_v$ is a constant map, where

$$f^+(u) = \sum_{(u,w)\in E} \sum_{i\in f(u.w)} i \pmod v$$

Thus, finding an edge-magic labeling of G[k] is equivalent to finding a k-fold edge-magic labeling of G.

The following example illustrates that $P_4[4]$ is edge-magic.

Example 3.

Consider the path P_4 and suppose $k = 4$. $[12] \equiv [4] \times 3 \pmod 4$.

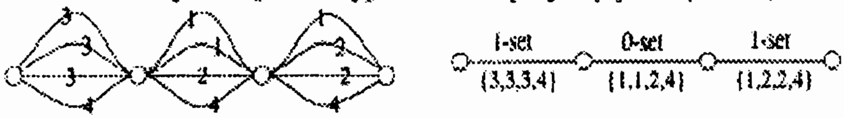


Figure 4.

Notation. For a cycle C_n with vertex set $\{x_1, x_2, \ldots, x_n\}$, we denote by $C_n(t)$ the cycle with a chord (x_1, x_t).

Example 4. The edge-magic index of $C_4(3)$ is 3. The 3-fold of $C_4(3)$ has 15 edges. We observe that $[15] \equiv [4] \times 3 \cup \{1, 2, 3\} \pmod 4$. Figure 5 depicts a 3-fold edge-magic labeling of $C_4(3)$.

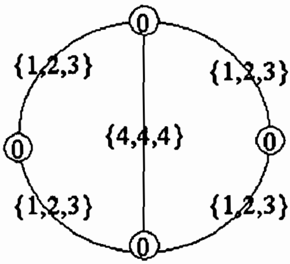


Figure 5.

We have the following general result.

Theorem 2. All (4, 5)-graphs have edge-magic index 3.

Proof. By Theorem 1, 3 is the smallest possible value. Partition the integers from 1 to 15 (mod 4) into 5 sets, each with 3 integers, namely, $\{1, 1, 2\} \times 2$, $\{2, 3, 3\} \times 2$ and $\{0, 0, 0\}$. Each of these sets is a 0-set (mod 4), giving a 3-fold edge-magic labeling.

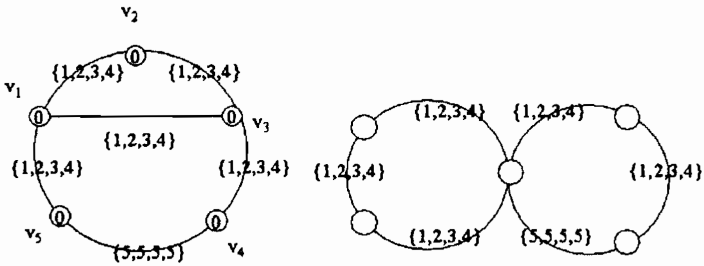
Note that the labeling in Theorem 2 is different from the one shown in Figure 5. In other words, there could be multiple edge-magic labelings.

<u>Theorem 3.</u> All (5, 6)-graphs have edge-magic index 4.
Proof. Suppose G is a (5, 6)-graph. By Theorem 1 we see that the edge-magic index is at least 4. Now $4 \times 6 = 24$ and $[24] = [4] \times 5 \cup \{0, 0, 0, 0\} \pmod 5$.

Consider the partition $\wp$ of [24]: $[4] \times 5$, $\{0, 0, 0, 0\}$. We see that each class is a 0-set. Therefore any bijection $f : E(G) \rightarrow \wp$ is a 4-fold edge-magic labeling.

These two Theorems will be generalized in Theorem 6 below.

<u>Example 5.</u> The following two (5, 6)-graphs have edge-magic index 4.



$C_5(3)$

Figure 6.

<u>Remark.</u> We want to note here that even if v divides k(k + 1), there is no guarantee that G[k] is edge-magic. In other words, the edge-magic index could be strictly greater than such k. The following graph provides an example.

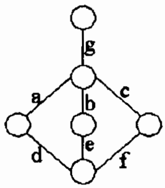


Figure 7.

Assume, on the contrary, that the edge-magic index = 2. Then there are 14 edges, with edge sum = $105 \equiv 3 \pmod 6$. Let the sum of the two parallel edge labels be a, b, c, d, e, f, g (mod 6), as shown in the above diagram. Then $a + b + c + d + e + f + g \equiv 3 \pmod 6$. Then, $g \equiv a + b + c + g \equiv a + d \equiv b + e \equiv c + f \equiv d + e + f \pmod 6$. It follows that $a + b + c \equiv 0$, $a \equiv e + f$, $b \equiv d + f$, $c \equiv d + e$. Adding the last three congruences, we have $a + b + c \equiv 2(d + e + f)$. Since $g \equiv d$

+ e + f (mod 6) and a + b + c ≡ 0 (mod 6), we have a + b + c + d + e + f + g ≡ 2(d + e + f) ≡ 0 (mod 6), which is a contradiction. The minimum k such that 6 divides k(k + 1) is 2. But min IM(G) is not 2.

We can show that the edge-magic index is not 3 by a similar argument. Assume not. Then G[3], with 21 edges, is edge-magic. The total edge sum = 231 ≡ 3 (mod 6). An argument identical to the above gives a contradiction.

Now we show that its edge-magic index = 5, by explicitly giving an edge-magic labeling (Figure 8). Note that this labeling will be discussed more generally in Theorem 5.

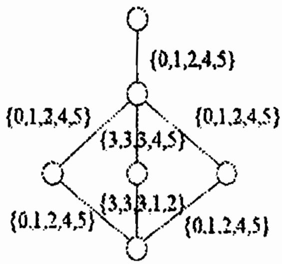
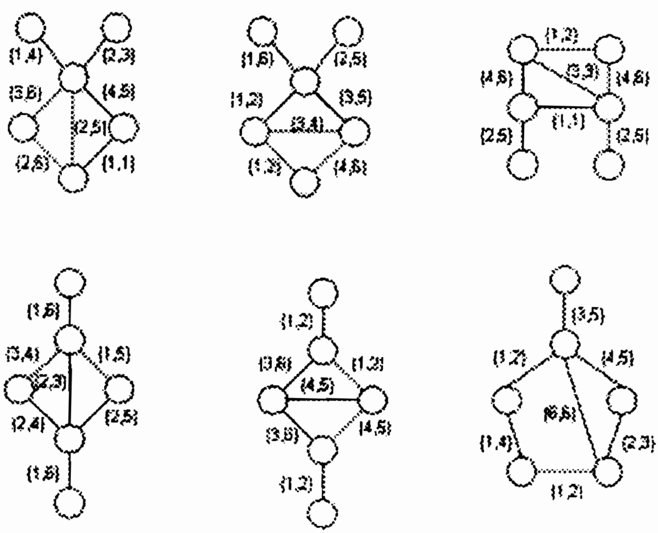


Figure 8.

<u>Theorem 4.</u> All connected (6, 7)-graphs, except the one depicted in Figure 8, have edge-magic indices 2.

Proof. Up to isomorphism, there are nineteen connected (6, 7)-graphs. We see that the edge-magic index is 2 for each of the following 18 graphs. (Figure 9)



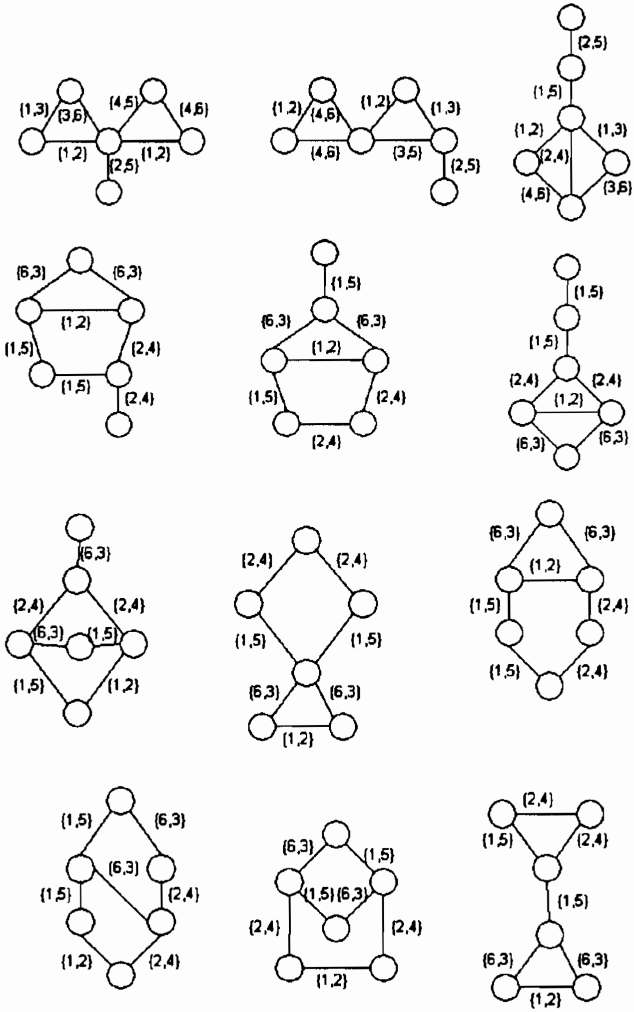


Figure 9.

__Theorem 5.__ A $(v, v + 1)$-graph G has edge-magic index at most $v - 1$.
Proof. It suffices to show that $G[v - 1]$, which has $q = v^2 - 1$ edges, is edge-magic.
Case 1: v is odd.
The case when $v = 1$ is trivial.

For $v > 1$, partition the integers from 1 to $v^2 - 1$ (mod v) into $v + 1$ sets, each with $v - 1$ integers, namely $[v - 1] \times v$, and $\{0, 0, \dots, 0\}$ (with $v - 1$ zeros). Each of these sets is a 0-set (mod v), giving a $(v - 1)$-fold magic labeling of G.

Case 2: v is even.

When $v = 2$, we have 2 vertices and 3 edges. $G[1] = G$ is obviously edge-magic.

For $v = 4$, see Theorem 2.

For $v = 6$, see Theorem 4 and the Remark before it.

Now consider $v \geq 8$. Partition the integers from 1 to $v^2 - 1$ (mod v) into $v + 1$ sets, each with $v - 1$ integers, namely $\{0, 1, 2, \dots, (v/2) - 1, (v/2) + 1, \dots, v - 1\} \times (v - 1)$, $(\{v/2\} \times (v - 3)) \cup \{1, (v/2) - 1\}$, and $(\{v/2\} \times 3) \cup \{2, \dots, (v/2) - 2, (v/2) + 1, \dots, v - 1\}$. Each of the first $(v - 1)$ sets has a sum of $v(v - 1)/2 - (v/2) = v(v - 2)/2 \equiv 0 \pmod v$. The second to last set has a sum of $(v/2)(v - 3) + (v/2) = v(v - 2)/2 \equiv 0 \pmod v$. The last set has a sum of $3(v/2) + v(v - 1)/2 - 1 - ((v/2) - 1) - (v/2) = (v/2)v \equiv 0 \pmod v$. Thus each of these sets is a 0-set (mod v), giving a $(v - 1)$-fold edge-magic labeling of G.

3. (v, v + 1)-graphs with v = a prime power.

Theorem 6. Let p be a prime and n a positive integer. The edge-magic index of a $(p^n, p^n + 1)$-graph is $p^n - 1$.

Proof. Consider two cases.

Case 1: p = 2.

Let the edge-magic index be k. The earlier necessary condition reduces to $2^n \mid k(k + 1)$. The smallest possible value of k satisfying this necessary condition is $2^n - 1$. It suffices to show that $G[2^n - 1]$ is edge-magic.

For $n = 1$, we have 2 vertices and 3 edges. $G[1] = G$ is obviously edge-magic.

For $n = 2$, see Theorem 2.

For $n \geq 3$, the proof of Theorem 5 gives such a labeling.

Case 2: p is odd.

Let the edge-magic index be k.

The earlier necessary condition reduces to $p^n \mid k(k + 1)$. The smallest possible value of k satisfying this necessary condition is $p^n - 1$. It suffices to show that $G[p^n - 1]$ is edge-magic. The proof of Theorem 5 gives such a labeling.

Corollary 7. All (8, 9)-graphs have edge-magic indices 7.

Example 6. Consider (8, 9)-graphs. The 0-sets (mod 8) are 7 copies of $\{0, 1, 2, 3, 5, 6, 7\}$, $\{4, 4, 4, 4, 4, 1, 3\}$, and $\{4, 4, 4, 2, 5, 6, 7\}$.

The edge-magic index of $C_8(5)$ is 7.

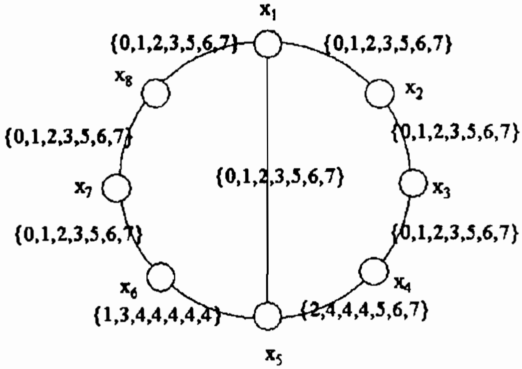


Figure 10.

Example 7. Consider (16, 17)-graphs. We observe $16 = 2^4$, i.e., n = 4. The 0-sets (mod 16) are 15 copies of {0, 1, 2, ... , 7, 9, ... , 15}, $(\{8\} \times 13) \cup \{1, 7\}$, and {8, 8, 8, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15}.

Example 8. The following (7, 8)-graphs have edge-magic indices 6.

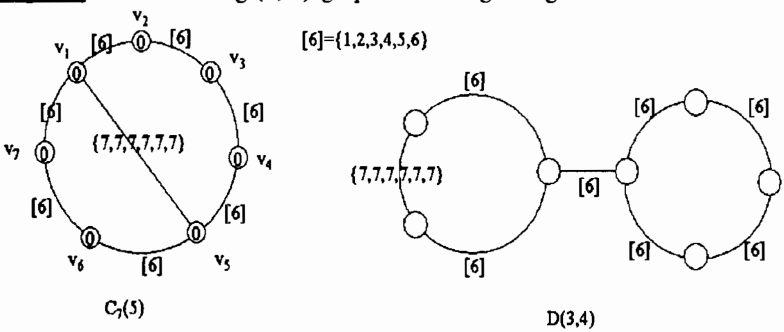


Figure 11.

We can consider all cases of (v, v + 1)-graphs settled for $v \leq 11$. Note that v = 2, 3, 4, 5, 7, 8, 9, and 11 are primes or prime powers, and so are covered by Theorem 6. Theorem 4 deals with the case when v = 6. For v = 10, Theorem 1 shows that the edge-magic index ≥ 4. Since $[44] = ([10] \times 4) \cup \{1, 2, 3, 4\}$, the following 0-sets show that the edge-magic index ≤ 4: {0, 0, 0, 0}, {5, 5, 5, 5}, {1, 2, 3, 4}, two copies of {1, 1, 9, 9}, two copies of {2, 2, 8, 8}, two copies of {3, 3, 7, 7}, and two copies of {4, 4, 6, 6}.

4. Two families of (v, v + 1)-graphs.

In this section we show that there exist infinitely many $(v, v+1)$-graphs with edge-magic index less than $v-1$. We also show that for certain values of v, there can be different $(v, v+1)$-graphs with different edge-magic indices.

Construction 1:

Let $v = 2n(2n+1)$, where n is a positive integer. Consider the graph $C_v(3)$, i.e., the cycle with v vertices, with an additional edge joining the first and third vertices. There are $v+1$ edges, namely, the triangle connecting the first, second, and third vertices, and $2n(2n+1)-2$ other edges in the cycle C_v. We will show that it has edge-magic index 2n.

Let the edge-magic index be k. The necessary condition earlier in this paper reduces to $2n(2n+1) \mid k(k+1)$. The smallest possible value of k satisfying this necessary condition is 2n. It suffices to show that $C_v(3)[2n]$ is edge-magic.

In $C_v(3)[2n]$, there are $2n(2n(2n+1)+1)$ edges. Mod v, the set of edge labels $= (\{1, 2, \dots, 2n(2n+1)-1\} \leftrightarrow 2n) \lceil (\{0\} \leftrightarrow 2n) \lceil \{1, 2, \dots, 2n\}$. For the $2n(2n+1)-2$ edges other than the three forming the triangle, use the following 0-sets to label them: $\{1, 2n(2n+1)-1\} \leftrightarrow n$, $\{2, 2n(2n+1)-2\} \leftrightarrow n, \dots, \{n(2n+1)-1, n(2n+1)+1\} \leftrightarrow n$, with each of them used twice. Label the three edges in the triangle using $(\{0\} \leftrightarrow (2n-1)) \lceil \{n(2n+1)\}$, $\{0\} \lceil (\{n(2n+1)\} \leftrightarrow (2n-1))$, $\{1, 2, \dots, 2n\}$, each of which is an $n(2n+1)$-set (mod v). This gives a (2n)-fold edge-magic labeling.

<u>**Example 9.**</u> Consider n = 1. We consider the (6, 7)-graph $C_6(3)$. The 2-fold edge-magic labeling is given by the 3-sets {0, 3}, {0, 3}, {1, 2} for the triangle, and the 0-sets {1, 5}, {1, 5}, {2, 4}, {2, 4} for the other edges. Note that this is illustrated in the 15th case of Figure 9.

<u>**Example 10.**</u> Consider n = 2. We consider the (20, 21)-graph $C_{20}(3)$. The 4-fold edge-magic labeling is given by the 10-sets {0, 0, 0, 10}, {0, 10, 10, 10}, {1, 2, 3, 4} for the triangle, and the 0-sets {1, 1, 19, 19}, {1, 1, 19, 19}, {2, 2, 18, 18}, {2, 2, 18, 18}, … , {9, 9, 11, 11}, {9, 9, 11, 11} for the other edges.

Construction 1 gives the following Theorem.

<u>**Theorem 8.**</u> There are infinitely many $(v, v+1)$-graphs with edge-magic index $< v-1$.

Construction 2:

Consider the following $(v, v+1)$-graph, where $v = 2n(2n+1)$, with n being an odd number. (Figure 12)

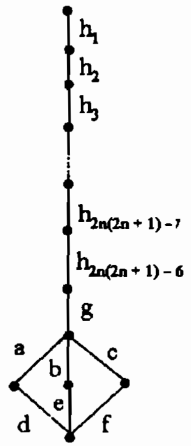


Figure 12.

We show that its edge-magic index is not 2n. Assume the contrary, i.e., its edge-magic index is 2n, and that the above diagram shows the sum of the parallel edge labels in G[2n].

Mod 2n(2n + 1), we have $h_1 \ldots h_1 + h_2 \ldots h_2 + h_3 \ldots h_3 + h_4 \ldots \ldots \ldots h_{2n(2n+1)-7} + h_{2n(2n+1)-6} \ldots h_{2n(2n+1)-6} + g \ldots g + a + b + c \ldots a + d \ldots b + e \ldots c + f \ldots d + e + f$. Thus $h_2 \ldots h_4 \ldots \ldots \ldots h_{2n(2n+1)-6} \ldots 0$, $h_1 \ldots h_3 \ldots h_5 \ldots \ldots \ldots h_{2n(2n+1)-7} \ldots g$, $a + b + c \ldots 0$, $a \ldots e + f$, $b \ldots d + f$, $c \ldots d + e$. From the last four congruences, we have $0 \ldots a + b + c \ldots 2(d + e + f)$. Since $g + a + b + c \ldots d + e + f$, we have $g \ldots d + e + f$. Thus the sum of all edge labels $= h_1 + h_2 + h_3 + h_4 + \ldots + h_{2n(2n+1)-7} + h_{2n(2n+1)-6} + g + a + b + c + d + e + f \ldots (n(2n + 1) - 2)g + d + e + f \ldots (n(2n + 1) - 1)(d + e + f)$. Since n is odd, $n(2n + 1) - 1$ is even, making the sum of all edge labels … 0.

On the other hand, the edge labels are $1, 2, \ldots, 2n(2n(2n + 1) + 1)$. The sum is $(1 + 2n(2n(2n + 1) + 1))n(2n(2n + 1) + 1) \ldots (1 + 2n)n$, which is not 0 (mod 2n(2n + 1)), giving a contradiction. Thus the edge-magic index cannot be 2n.

Example 11. The graph in Figure 7 corresponds to the case n = 1 in this construction.

Constructions 1 and 2 give the following Theorem.

Theorem 9. There exist infinitely many values v such that for each such v, there exist two (v, v + 1)-graphs with different edge-magic indices.

5. Further work.

We might think that almost all $(v, v + 1)$-graphs have edge-magic index $v - 1$, in the sense that if $A(v) = |(v, v + 1)$-graphs having $v - 1$ as their edge-magic indices$|$ and $B(v) = |$ all connected $(v, v + 1)$-graphs$|$, then $\lim A(v)/B(v) = 1$, as $v \to \infty$.

We will exhibit two classes of graphs to show that this conjecture is not true.

Consider $v = 2m$, where m is an odd number of the form $4k + 1$, i.e., $v = 8k + 2$. We show that all such $(v, v + 1)$ graphs have edge-magic index $\leq m - 1 < v - 1$. Note that $[(m - 1)(v + 1)] = ([v] \times (m - 1)) \cup [m - 1]$, where $1 + 2 + \ldots + (m - 1) = ((m - 1)/2)m = 2km \equiv 0 \pmod v$. Label the $(2m + 1)$ edges with the following $(0; m - 1)$-sets: $\{0, 0, \ldots, 0\}$, $\{m, m, \ldots, m\}$, $\{1, 2, \ldots, m - 1\}$, two copies of $(\{1\} \times ((m - 1)/2)) \cup (\{v - 1\} \times ((m - 1)/2))$, two copies of $(\{2\} \times ((m - 1)/2)) \cup (\{v - 2\} \times ((m - 1)/2))$, ... , and two copies of $(\{m - 1\} \times ((m - 1)/2)) \cup (\{m + 1\} \times ((m - 1)/2))$. Thus at least 1/8 of the $(v, v + 1)$-graphs have edge-magic index $< v - 1$.

Now consider $v = 2m$, where m is an odd number of the form $4k + 3$, i.e., $v = 8k + 6$. If the graph contains a triangle, we can modify our technique in Construction 1 to show that such graphs have edge-magic index $\leq m - 1 < v - 1$. Note that $[(m - 1)(v + 1)] = ([v] \times (m - 1)) \cup [m - 1]$, where $1 + 2 + \ldots + (m - 1) = ((m - 1)/2)m = (2k + 1)m \equiv m \pmod v$. Label the 3 edges of the triangle using $(\{0\} \times (m - 2)) \cup \{m\}$, $\{0\} \cup (\{m\} \times (m - 2))$, and $\{1, 2, \ldots, m - 1\}$. Note that each of these is an $(m; m - 1)$-set. Label the other $(2m - 2)$ edges using two copies of $(\{1\} \times ((m - 1)/2)) \cup (\{v - 1\} \times ((m - 1)/2))$, two copies of $(\{2\} \times ((m - 1)/2)) \cup (\{v - 2\} \times ((m - 1)/2))$, ... , and two copies of $(\{m - 1\} \times ((m - 1)/2)) \cup (\{m + 1\} \times ((m - 1)/2))$. Note that each of these is an $(0; m - 1)$-set. Thus these $(v, v + 1)$-graphs have edge-magic index $< v - 1$.

We end this paper with a problem: What is the limit of $A(v)/B(v)$, as $v \to \infty$?

Acknowledgement. We express our thanks to the referee for his helpful suggestions.

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