

On the Supermagic Edge-splitting Extension of Graphs

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ABSTRACT. A (p,q) -graph G in which the edges are labeled $1,2,3,\dots,q$ so that the vertex sums are constant, is called supermagic. If the vertex sum mod p is a constant, then G is called edge-magic. We investigate the supermagic characteristic of a simple graph G , and its edge-splitting extension $SPE(G,f)$. The construction provides an abundance of new supermagic multigraphs.

AMS 2000 MSC: 05C78, 05C25

1. Introduction. Magic graphs were first initiated by Sedlacek (around 1963) [14,15] as the problem of labeling the edges of the graph with real numbers so that the sum of the edges label to a vertex is same as all the other vertices. Jeurissen [5], Jezney and Trenkler [6] gave characterizations of magic graphs. A characterization of regular magic graphs in term of even circuits is given by Doob [1]. Kong, Sun, Lee et al [7,24,25] provided some general constructions of magic graphs. Since Sedlacek's original article [14], literally hundred of papers have been written on magic graph labelings (see the survey article [2]).

If G is a (p,q) -graph in which the edges are labeled $1,2,3,\dots,q$ so that the vertex sums defined by $f^+(u) = \sum\{f(u,v) : (u,v) \in E\}$ is constant, then G is called **supermagic**. Figure 1 shows a graph with 6 vertices and 8 edges which is supermagic.

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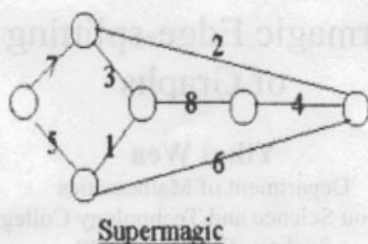


Figure 1

B.M. Stewart [22,23] showed that K_3, K_4, K_5 are not supermagic and when $n \equiv 0 \pmod{4}$, K_n is not supermagic. For $n > 5$, K_n is supermagic if and only if $n \not\equiv 0 \pmod{4}$. Hartsfield and G. Ringel [2] provided some classes of supermagic graphs. Ho and Lee [4] extended the result of Stewart to regular complete k-partite graphs. Recently Shiu, Lam and Cheng [17] considered a class of supermagic graphs which are disjoint union of $K_{3,3}$.

A generalization of supermagic graphs was introduced by Lee, Seah and Tan [10]. A graph $G = (V, E)$ with p vertices and q edges is called **edge-magic** if there is a bijection $f : E \rightarrow \{1, 2, \dots, q\}$ such that the induced mapping $f^+ : V \rightarrow \mathbb{Z}_p$, given by $f^+(u) = \sum \{f(u,v) : (u,v) \in E\} \pmod{p}$ is a constant. A necessary condition for a (p,q) -graph to be edge-magic is $q(q+1) \equiv 0 \pmod{p}$. However, there are infinitely many connected graphs such as trees, cycles satisfy the necessary condition but are not edge-magic.

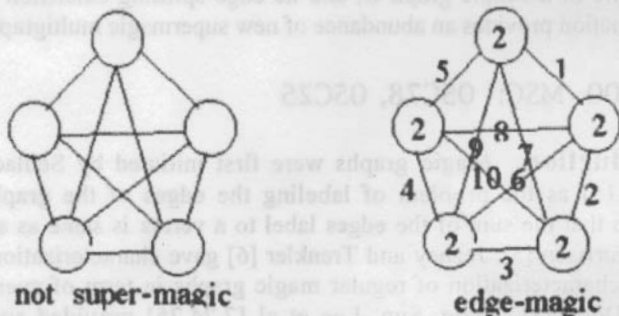


Figure 2

The concept of edge-magic labeling of graphs is the dual concept of edge-graceful labeling [10]. In 1985, Lo Sheng-Ping [13] introduced the concept of edge-graceful graphs. A (p,q) -graph $G = (V, E)$, of p vertices and q edges, is said to be **edge-graceful** if there exists a bijection $f : E \rightarrow \{1, 2, \dots, q\}$ such that the induced mapping $f^+ : V \rightarrow \{0, 1, \dots, p-1\}$, defined by $f^+(v) = \sum \{f(u,v) : (u,v) \in E(G)\} \pmod{p}$ is a bijection.

The cartesian product of two paths is frequently called the grid graph. The cartesian product of two cycles is called the torus graph. It was shown in [16] that the torus graph $C_m \times C_n$ is edge-magic for all $m, n > 2$.

Karl Schaffer and Sin-Min Lee [16] have shown that if G and H are both odd-order, regular, edge-graceful graphs, where G is d -regular and has m vertices, and H is k -regular and has n vertices, and furthermore $\text{GCD}(d, n) = \text{GCD}(k, m) = 1$, then $G \times H$ is edge-graceful. In particular, they showed that the torus graph $C_{2i+1} \times C_{2j+1}$ is edge-graceful.

Finding the magic labeling of graphs are related to solving system of linear Diophantine equations [21]. In general it is difficult to find an edge-magic or supermagic labeling of a graph. Several classes of graphs had been shown to be edge-magic ([8,9,10,11,12,17,18]). For more conjectures and open problems on edge-magic graphs the reader is referred to [8,9,10,11,12].

Shiu, Lam and Lee [18,19] give a general construction of supermagic graphs and edge-magic graphs. The reader should see the survey article of Gallian [2] for various labeling problems.

In this paper we introduce a construction of super magic graphs by splitting some of the edges of the graphs. We consider this construction for perfect matchings and cycles.

2. Edge-splitting extensions of graphs.

In this section, we shall introduce a general construction of multigraphs from a given simple graph. Let $N = \{1, 2, 3, \dots\}$ be a set of natural numbers.

Given a pair (G, f) where $G = (V, E)$ is a simple graph with p vertices and q edges and $f: E(G) \rightarrow N$, we can construct a graph $\text{SPE}(G, f)$ as follows:

For each edge e of $E(G)$ if $f(e) = k$, we associate a set of parallel edges $P(e) = \{e_1, e_2, \dots, e_k\}$. We observe that $V(\text{SPE}(G, f)) = V(G)$ and $E(\text{SPE}(G, f)) = \cup \{P(e) : e \in E(G)\}$

We shall call the graph $\text{SPE}(G, f)$ as an **edge-splitting extension graph** of (G, f) .

We illustrate here with one example:

Example 1. Let $G = C_4$ and $f: E(C_4) \rightarrow N$ as follows:

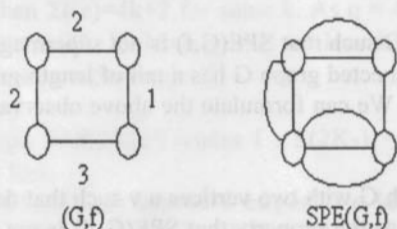


Figure 3

A necessary condition for a (p,q) -graph to be supermagic is $q(q+1) \equiv 0 \pmod{p}$. However, this is not the sufficient condition. Consider the following example:

Example 2. Let $G = C_6$ and $h: E(C_6) \rightarrow \mathbb{N}$ be defined as follows:

	a	b	c	d	e	f
h	1	2	1	2	1	2

Then we see that the following $SPE(C_6, h)$ (Figure 4) has 6 vertices and 9 edges.

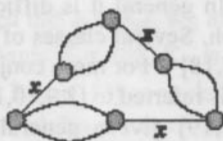


Figure 4

If the graph is supermagic then three single edges must be labeled by x for some x which is not possible.

Remark. In fact, if G is a graph with three edges a , b , and c such that a is adjacent to b and b is adjacent to c but a and c are not adjacent and $f: E(G) \rightarrow \mathbb{N}$ with property $f(a) = f(c) = 1$, then $SPE(G, f)$ is not supermagic.

Theorem 1. If G is a regular simple graph and $f: E(G) \rightarrow \mathbb{N}$ is a constant map with $f(e) = 2t$ for a fixed t , then $SPE(G, f)$ is supermagic.

Proof. Suppose G is a (p, q) -graph. We see that $SPE(G, f)$ has $2tq$ edges. We divide the set of integers $\{1, 2, \dots, 2tq\}$ into tq pairs $\{1, 2tq\}, \{2, 2tq-1\}, \dots, \{k, 2tq-k+1\}, \dots, \{tq, tq+1\}$. We denote this set by H

For each e of G , $P(e) = \{e_1, e_2, \dots, e_{2t}\}$ we form the t pairs $\{e_1, e_2\}, \{e_3, e_4\}, \dots, \{e_{2t-1}, e_{2t}\}$.

Then the set of edges $SPE(G, f)$ has tq pairs. We denote the set by Q . Any bijection $F: Q \rightarrow H$ induces a super edge-magic labeling $g: E(SPE(G, f)) \rightarrow \{1, 2, \dots, 2tq\}$ by letting $g(a) = i$ and $g(b) = j$ if $F(\{a, b\}) = \{i, j\}$. \square

There exists graph G such that $SPE(G, f)$ is not supermagic for any $f: E(G) \rightarrow \mathbb{N}$. For example a connected graph G has a tail of length greater than or equal to one has this property. We can formulate the above observation by the following result.

Theorem 2. Any graph G with two vertices u, v such that $\deg(u) = 1$ and $\deg(v) \geq 2$ and u, v are adjacent has the property that $SPE(G, f)$ is not supermagic for any $f: E(G) \rightarrow \mathbb{N}$.

Corollary 3. For any $n > 1$, the complete bipartite graph $K_{1,n}$ has the property that $SEP(K_{1,n}, f)$ is not supermagic for any $f: E(G) \rightarrow \mathbb{N}$.

Corollary 4. If $n > 2$, then the graph $\text{SEP}(P_n, f)$ is not supermagic for all $f: E(G) \rightarrow N$.

Theorem 5.(a) The graph $G=P_2$ has the property that $\text{SPE}(G, f)$ is supermagic for any $f: E(G) \rightarrow N$. \square

(b) P_2 is the only graph G with the property that $\text{SPE}(G, f)$ is supermagic for any $f: E(G) \rightarrow N$. \square

3. Edge-splitting extension of perfect matchings.

Let mK_2 be the m perfect matching and $f: E(mK_2) \rightarrow N$ with $f(e) = n$ for all e in $E(G)$.

Shiu and Lee [20] showed that:

Theorem 6. For $m, n \geq 2$, the splitting-edge extension of the m perfect matching and $f: E(G) \rightarrow N$ with $f(e) = n$ for all e in $E(G)$ is supermagic if and only if n is even or both m and n are odd.

Example 3. We label $\text{SPE}(3K_2, f)$ where $f: E(3K_2) \rightarrow N$ with $f(e) = 3$ for all e of $3K_2$ as follows:

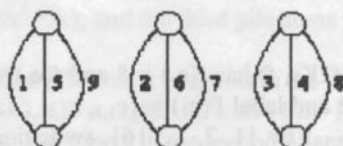


Figure 5

Theorem 7. If the graph $\text{SPE}(2K_2, f)$ where $f: E(2K_2) \rightarrow N$ is supermagic, then $\sum f(e) \not\equiv 1 \pmod{4}$ or $\sum f(e) \not\equiv 2 \pmod{4}$.

Proof. If $\sum f(e) \equiv 1 \pmod{4}$ then $\sum f(e) = 4k+1$ for some k . As $q = 4k+1$, we have $q(q+1) = (4k+1)(4k+2) = 2(4k+1)(2k+1)$.

Now $p=4$. It is clear that $q(q+1) \not\equiv 0 \pmod{4}$. Thus $\text{SPE}(2K_2, f)$ cannot be supermagic.

If $\sum f(e) \equiv 2 \pmod{4}$ then $\sum f(e) = 4k+2$ for some k . As $q = 4k+2$, we have $q(q+1) = (4k+2)(4k+3) = 2(4k+3)(2k+1)$. Now $p=4$. It is clear that $q(q+1) \not\equiv 0 \pmod{4}$. Thus $\text{SPE}(2K_2, f)$ cannot be supermagic. \square

Theorem 8. The graph $\text{SPE}(2K_2, f)$ where $f: E(2K_2) \rightarrow N$ is supermagic if $f(a) = 2k-1$, $f(b) = 2k$ for $k \geq 1$.

Proof. We see that $\text{SPE}(2K_2, f)$ has $4k-1$ edges. We label the edges $P(a) = \{a_1, a_2, \dots, a_{2k-1}\}$ by $A = \{1, 2, 3, \dots, [(2k-1)/2], 4k-1, 4k-2, \dots, 4k - [(2k-1)/2] - 1\}$. The other edges $P(b) = \{b_1, b_2, \dots, b_{2k}\}$ will be labeled by the complement of A in $\{1, 2, \dots, 4k-1\}$.

We observe that this is a supermagic labeling. \square

Example 4. We can have a supermagic labeling of $SPE(2K_2, f)$ with $f(a)=7, f(b)=8$ as follows:

$$\{1,2,3, 15,14,13,12\}, \{4,5,6,7,8,9,10,11\}.$$

Theorem 9. The graph $SPE(2K_2, f)$, where $f : E(2K_2, f) \rightarrow \mathbb{N}$, is supermagic if $f(e_1) = n$, $f(e_2) = 2n$ for $n \equiv 0 \pmod{4}$ or $n \equiv 1 \pmod{4}$.

Proof. We consider two cases:

Case 1. $n \equiv 0 \pmod{4}$. Let $n = 4k$, $k \geq 1$.

The sum of the first $8k$ integers is $4k(8k+1)$, and the sum of the last $4k$ integers is $2k(20k+1)$. Since they differ by $8k^2 - 2k = (k-1)8k + 6k$. We switch $\{7k + 2, 7k + 3, \dots, 8k\}$ with $\{11k + 2, 11k + 3, \dots, 12k\}$ and $10k$ with $7k$. For the $k-1$ pairs of swap, the first sum increases by $4k$ and the second sum decreases by $4k$, while the last pair increases the first sum by $3k$ and decreases the second sum by $3k$. As a result, they both sum up to $3k(12k + 1) = 3n(3n + 1)/4$. Therefore $SPE(2K_2, f)$ is supermagic.

Case 2. $n \equiv 1 \pmod{4}$. Let $n = 4k+1$, $k \geq 1$.

The sum of the first $8k + 2$ numbers is $(4k + 1)(8k + 3)$, and the last $4k + 1$ numbers sum up to be $(4k + 1)(10k + 3)$. They differ by $2k(4k + 1)$; switching the last k numbers will give both sums $(4k + 1)(9k + 3)$. We see that $SPE(2K_2, f)$ is supermagic. \square

Example 5. (a) If $SPE(2K_2, f)$ has $f(e_1) = 8$ and $f(e_2) = 16$, we follow case 1 of Theorem 10, with $k = 2$ and label $P(e_1) = \{e_{1,1}, e_{1,2}, \dots, e_{1,8}\}$ by $\{17, 18, \dots, 24\}$ $P(e_2) = \{e_{2,1}, e_{2,2}, \dots, e_{2,16}\}$ by $\{1, 2, \dots, 16\}$, switching 24 with 16 and 20 with 14, we have a supermagic labeling.

(b) If $SPE(2K_2, f)$ has $f(e_1) = 13$ and $f(e_2) = 26$, we follow case 2 of Theorem 10, with $k = 3$ and label $P(e_1) = \{e_{1,1}, e_{1,2}, \dots, e_{1,13}\}$ by $\{27, 28, \dots, 39\}$ $P(e_2) = \{e_{2,1}, e_{2,2}, \dots, e_{2,26}\}$ by $\{1, 2, \dots, 26\}$, switching 39 with 26, 38 with 25, and 37 with 24, we have a supermagic labeling.

Remark. When $m = 3$ and $f : E(3K_2) \rightarrow \mathbb{N}$ satisfies the necessary condition of supermagicness, that is $(\sum f(e)) (1 + \sum f(e)) \equiv 0 \pmod{6}$, we do not guarantee that $SPE(3K_2, f)$ is supermagic.

For example if $E(3K_2) = \{a, b, c\}$ and we consider the following two mappings:

(1) $f(a)=1$, $f(b)=2$ and $f(c)=2$. We see that $5 \times 6 \equiv 0 \pmod{6}$. The graph $SPE(3K_2, f)$ is supermagic with the label $\{5\}, \{1,4\}$ and $\{2,3\}$.

(2) $f(a)=1$, $f(b)=2$ and $f(c)=3$. We see that $6 \times 7 \equiv 0 \pmod{6}$. If $SPE(3K_2, f)$ is supermagic then the vertex sum should be 7 which is not possible to have such a labeling.

Theorem 10 A necessary condition for $SPE(3K_2, f)$ with $f : E(3K_2) \rightarrow \mathbb{N}$, $f(e_1) = f(e_2) = n$, $f(e_3) = k$ to be supermagic is $q(q+1) \equiv 0 \pmod{6}$ and $q(q+1)/3k \leq 2q+1 - k$.

Proof. Suppose $f(e_1) = f(e_2) = n$, $f(e_3) = k$ and $\text{SPE}(3K_2, f)$ is supermagic. Let $A = \{x_1, x_2, \dots, x_n\}$, $B = \{y_1, y_2, \dots, y_n\}$, $C = \{z_1, z_2, \dots, z_k\}$ be the edge labels of $\text{SPE}(3K_2, f)$. Then we have

$$\sum x_i = \sum y_j = \sum z_m = q(q+1)/6.$$

Since the sum of $\{q, (q-1), \dots, q-k+1\}$ is $(k/2)(2q+1-k)$, thus $q(q+1)/6 \leq (k/2)(2q+1-k)$, i.e., $q(q+1)/3k \leq 2q+1-k$. \square

Remark. The above necessary condition for $\text{SPE}(3K_2, f)$ to be supermagic is not sufficient in general.

Consider $\text{SPE}(3K_2, f)$ with $f(e_1) = f(e_2) = 2$, $f(e_3) = 7$. Here $q = 11$ and $q(q+1)/6 = 22$. We observe that $q(q+1)/3k \leq 2q+1-k$.

However, it is impossible to find two numbers from $\{1, 2, \dots, 11\}$ whose sum is 22; hence, $\text{SPE}(3K_2, f)$ is not supermagic. \square

Theorem 11. $\text{SPE}(3K_2, f)$ is supermagic if $f: E(3K_2) \rightarrow \mathbb{N}$ is $f(e_1) = f(e_2) = n$, $f(e_3) = 2n$, for $n \equiv 0 \pmod{3}$ or $n \equiv 2 \pmod{3}$.

Proof. Case 1. $n \equiv 0 \pmod{3}$. Let $n = 3k$. Break the set $\{1, 2, \dots, 12k\}$ into 3 piles: $\{1, 2, \dots, 6k\}$, $\{6k+1, 6k+2, \dots, 9k\}$, and $\{9k+1, 9k+2, \dots, 12k\}$. The first pile sums up to $3k(6k+1) = 18k^2 + 3k$; the second pile sums up to $(3k/2)(15k+1) = (1/2)(45k^2+3k)$; and the third pile sums up to $(3k/2)(21k+1) = (1/2)(63k^2+3k)$.

Switch k numbers in the first pile with k numbers in the third pile with pairwise difference $6k-1$; e.g., $\{4k+2, 4k+3, \dots, 5k+1\}$ with $\{10k+1, 10k+2, \dots, 11k\}$.

Next if k is odd, we switch k numbers in the second pile with k numbers in the third pile with pairwise difference $(3k+1)/2$; e.g., $\{7k+(k+1)/2, 7k+(k+3)/2, \dots, 8k+(k-1)/2\}$ with $\{9k+1, 9k+2, \dots, 10k\}$.

If k is even, we simply switch any $k/2$ numbers of the second pile with the $k/2$ numbers of the third pile which differ pairwise by $3k+1$; e.g., $\{8k+(k/2), \dots, 9k-1\}$ with $\{11k+(k/2)+1, \dots, 12k\}$. Now all three piles sum up to $24k^2+2k$.

Case 2. For $n \equiv 2 \pmod{3}$, let $n = 3k+2$. We break the set $\{1, 2, \dots, 12k+8\}$ into three piles as before, $\{1, 2, \dots, 6k+4\}$, $\{6k+5, 6k+6, \dots, 9k+6\}$, $\{9k+7, \dots, 12k+8\}$.

The first pile sums up to $(3k+2)(6k+5) = 18k^2+27k+10$, the second pile sums up to $(1/2)(3k+2)(15k+11)$, and the third pile sums up to $(1/2)(3k+2)(21k+15)$.

We first switch k numbers in the first pile with the k numbers in the third pile with pairwise difference $6k$, and an extra pair with difference $7k+2$; e.g., $\{5k+5, \dots, 6k+4; 2k+5\}$ with $\{11k+5, \dots, 12k+4; 9k+7\}$.

If k is odd, we switch $(k+1)/2$ numbers in the second pile with the $(k+1)/2$ numbers in the third pile with pairwise difference $3k+2$; e.g., $\{7k+1, \dots, 7k+(k+1)/2\}$ with $\{10k+3, 10k+4, \dots, 10k+(k+1)/2+2\}$.

If k is even, we simply switch any $k/2$ numbers in the second pile with the $k/2$ numbers in the third pile with pairwise difference $3k$ and one pair with difference $(5k+2)/2$.

Then all three piles sum up to $(6k+4)(4k+3)$. \square

Example 6. If $n = 14$, $f(e_1) = f(e_2) = 14$, and $f(e_3) = 28$, we label e_1 with $\{29, 30, \dots, 37, 45, 46, \dots, 49\}$, e_2 with $\{1, 26, 27, 28, 38, 39, 40, 41, 42, 44, 50, 51, 52, 53\}$, and e_3 with $\{2, 3, \dots, 25, 43, 54, 55, 56\}$.

Now all three piles sum up to 532.

Remark. $\text{SPE}(3K_2, f)$ is not supermagic if $f: E(3K_2) \rightarrow N$ is $f(e_1) = f(e_2) = n$, $f(e_3) = 2n$, and $n \equiv 1 \pmod{3}$. For if $n = 3k+1$, then as $q = 4n = 12k+4$. We see that the $q(q+1) = 4(3k+1)(12k+5) \not\equiv 0 \pmod{6}$. Therefore it is not supermagic.

4. Edge-splitting extensions of cycles

In this section we investigate supermagicness of edge-splitting extensions of cycles.

Theorem 12. If m is odd and $k \geq 1$, then the cycle C_m , has supermagic edge splitting extension for all $f_n: E(C_m) \rightarrow N$ with $f_n(e) = n$ for all e in C_m .

For even n , Theorem 12 is a corollary of Theorem 1. For odd n , it is sufficient to prove for $n = 3$, a simple induction will prove for all odd $n > 3$.

We need the following Lemmas in the system of Diophantine equations:

Lemma 4.1. Let m be odd and let $s = 3(3m+1)/2$. A system of m Diophantine equations

$$X_{1,1} + X_{2,1} + X_{3,1} = s$$

$$X_{1,2} + X_{2,2} + X_{3,2} = s$$

$$X_{1,3} + X_{2,3} + X_{3,3} = s$$

.....

$$X_{1,m} + X_{2,m} + X_{3,m} = s$$

has a solution with distinct $X_{i,j}$ in $\{1, 2, \dots, 3m\}$.

Proof. Set $X_{1,1} = 1, X_{1,2} = 2, X_{1,3} = 3, \dots, X_{1,k} = k, \dots, X_{1,m} = m; X_{2,m} = m+1, X_{2,m-2} = m+2, \dots, X_{2,m-2k} = m+1+k, \dots, X_{2,1} = m + (m+1)/2$ and $X_{2,m-1} = m + (m+3)/2, X_{2,m-3} = m + (m+5)/2, \dots, X_{2,2} = 2m$.

The sums in each row are now distinct consecutive integers $m + (m+3)/2, m + (m+5)/2, \dots, 2m + (m+1)/2$.

Thus setting $X_{3,m-1} = 2m+1, X_{3,m-3} = 2m+2, \dots, X_{3,2} = 2m + (m-1)/2; X_{3,m} = 2m + (m+1)/2, X_{3,m-2} = 2m + (m+3)/2, \dots, X_{3,1} = 3m$.

We see that the sum of each line yields $4m + (m+3)/2 = 3(3m+1)/2$. \square

Lemma 4.2. Let $m \geq 4$ be even, and let $s = (9m+2)/2$. Then the system of m Diophantine equations

$$X_{1,1} + X_{2,1} + X_{3,1} = s$$

$$X_{1,2} + X_{2,2} + X_{3,2} = s+1$$

$$X_{1,3} + X_{2,3} + X_{3,3} = s$$

.....

$$X_{1,m} + X_{2,m} + X_{3,m} = s+1$$

has a solution in distinct integers in $\{1, 2, \dots, 3m\}$.

Proof. Set $X_{1,1} = 1, X_{1,2} = 2, X_{1,3} = 3, \dots, X_{1,k} = k, \dots, X_{1,m} = m; X_{2,m-1} = m+1, X_{2,m-3} = m+2, \dots, X_{2,1} = m + (m/2), X_{2,m} = m + (m/2) + 1, X_{2,m-2} = m + (m/2) + 2, \dots, X_{2,2} = 2m$.

The sums of each row are now $m + (m/2) + 1, m + (m/2) + 2, \dots, 2m + (m/2)$. They are consecutive and distinct, except $2m + 1$ does not occur. By setting $X_{3,m} = 2m + 1, X_{3,m-2} = 2m + 2, \dots, X_{3,2} = 2m + (m/2); X_{3,m-1} = 2m + (m/2) + 1, X_{3,m-3} = 2m + (m/2) + 2, \dots, X_{3,1} = 3m$.

We see that the odd rows sum up to be $4m + (m/2) + 1 = s$, and the even rows $4m + (m/2) + 2 = s + 1$. \square

Proof of Theorem 12. Let $E(C_m) = \{e_1, e_2, \dots, e_m\}$ and for each k between 1 and m , let $\{e_{k,1}, e_{k,2}, \dots, e_{k,n}\}$ be the parallel edges of $SPE(C_m, f)$ correspond to e_k .

We define the edge labeling $g: SPE(C_m, f) \rightarrow \{1, 2, \dots, mn\}$ by setting $g(e_{k,j}) = X_{k,j}$ for each k between 1 and m and j between 1 and 3.

Then by Lemma 4.1 and Lemma 4.2, it is a supermagic labeling.

For $n \geq 5$, we label the edges $3m + 1, 3m + 2, \dots, 4m$ consecutively in the counterclockwise manner; then label the next cycle $4m + 1, 4m + 2, \dots, 5m$ in the clockwise manner, with the edge labels $4m$ and $4m + 1$ having the same vertices.

This process is repeated until all the edges are labeled.

It is easy to see the labeling is supermagic.

Example 7. We give a supermagic labeling of $SPE(C_7, f_3)$. We set $X_{1,1} = 1, X_{1,2} = 2, X_{1,3} = 3, X_{1,4} = 4, X_{1,5} = 5, X_{1,6} = 6, X_{1,7} = 7, X_{2,1} = 11, X_{2,3} = 10, X_{2,5} = 9, X_{2,7} = 8, X_{2,2} = 14, X_{2,4} = 13, X_{2,6} = 12$.

Then we set $X_{3,2} = 17, X_{3,4} = 16, X_{3,6} = 15, X_{3,1} = 21, X_{3,3} = 20, X_{3,5} = 19, X_{3,7} = 18$, then we observe that each vertex has sum $2s = 2 \times 33 = 66$ (see Figure 6):

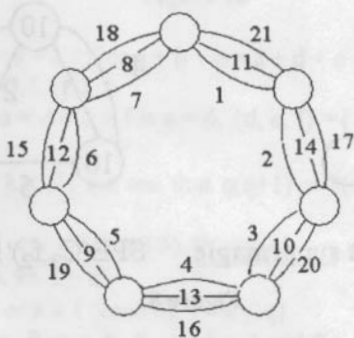


Figure 6

Example 8. We give a supermagic labeling of $SPE(C_5, f_5)$, we see that

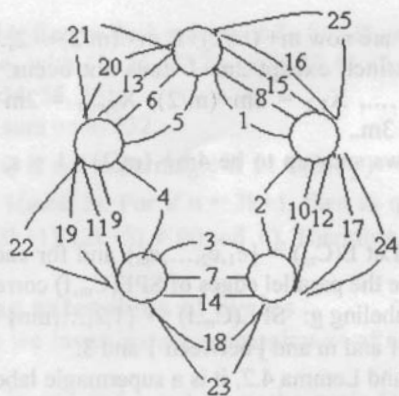
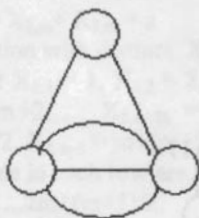


Figure 7

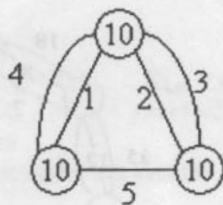
Example 9. Let $G = C_3$ and $f_i : E(C_3) \rightarrow \mathbb{N}$ for $i = 1, 2$ be defined as follows:

	a	b	c
f_1	1	1	3
f_2	2	2	1

Then we see that $\text{SPE}(C_3, f_1)$ is not supermagic. However, $\text{SEP}(C_3, f_2)$ is supermagic (see Figure 8).



$\text{SPE}(C_3, f_1)$ is not supermagic



$\text{SPE}(C_3, f_2)$ is supermagic

Figure 8

Theorem 13. Suppose $f(a) = f(b) = n$ and $f(c) = 1$ in C_3 then $\text{SPE}(C_3, f)$ is supermagic if and only if $n = 2$.

Theorem 14 Suppose $f(a) = f(b) = n$ and $f(c) = 2$ in C_3 , then $\text{SPE}(C_3, f)$ is supermagic if and only if $n = 2$ or 3 .

Proof. If $n = 2$ or 3 , the supermagicness of $\text{SPE}(C_3, f)$ is shown as Figure 9

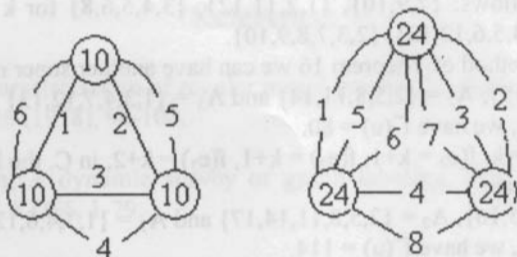


Figure 9

Theorem 15. Suppose $f(e_1) = k$, $f(e_2) = k+1$ and $f(e_3) = k+2$ in C_3 , then $SPE(C_3, f)$ is (a) not super magic for $k = 1$. (b) supermagic for $k \geq 2$.

Proof. (a) If $k = 1$, $SPE(C_3, f)$ has the following configuration (see Figure 10):

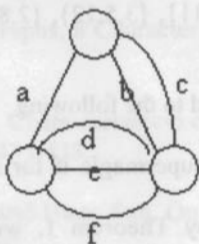


Figure 10

We see that $a = b + c = d + e + f$ or $a + b + c = a + d + e + f = b + c + d + e + f$ has no solution in $\{1, 2, 3, 4, 5, 6\}$.

The only way to have $a = d + e + f$ is $a = 6$, $\{d, e, f\} = \{1, 2, 3\}$ but there is no way to obtain b and c .

(b) Assume now that $k \geq 2$, we see that $q(q+1) = 3(n+1)(3n+4) = 0 \pmod{3}$. Now we set

$$C_1 = \{x: x \equiv 4 \pmod{6} \text{ or } x \equiv 3 \pmod{6}, 9 < x \leq q\}$$

$$C_2 = \{y: y \equiv 2 \pmod{3}, 9 < x \leq q\}$$

$$C_3 = \{z: z \equiv 0 \pmod{6} \text{ or } z \equiv 1 \pmod{6}, 9 < x \leq q\}$$

If k is even and $k = 2m$, $f(e_1) = k$, $f(e_2) = k + 1$ and $f(e_3) = k + 2$.

Let $A_1 = \{6, 9\} \cup C_1$, $A_2 = \{2, 5, 8\} \cup C_2$, $A_3 = \{1, 3, 4, 7\} \cup C_3$ be the edge labels for parallel edges of e_1, e_2, e_3 respectively.

For any u in C_3 we have $f^*(u) = 2(3m+2)(2m+1)$.

If k is odd $k = 2m+1$, we set $B_1 = \{7, 9\} \cup C_1$, $B_2 = \{2, 5, 8\} \cup C_2$, $B_3 = \{1, 3, 4, 6\} \cup C_3$ as the edge labels for parallel edges of e_1, e_2, e_3 respectively.

For all u in C we have $f^*(u) = 2(m+1)(6m+7)$.

Thus when $k \geq 2$, $SPE(C_3, f)$ is supermagic. \square

Example 10. We label $SPE(C_3, f)$ for $k = 2$ as follows: $\{7, 8\}$, $\{2, 4, 9\}$, $\{1, 3, 5, 6\}$;

for $k = 3$ as follows: $\{7,9,10\}$, $\{1,2,11,12\}$, $\{3,4,5,6,8\}$ for $k = 4$ as follows: $\{1,10,14,15\}$, $\{4,5,6,12,13\}$, $\{2,3,7,8,9,10\}$.

If we use the method of Theorem 16 we can have another super magic labeling:

$$A_1 = \{6,9,10,15\}, A_2 = \{2,5,8,11,14\} \text{ and } A_3 = \{1,3,4,7,12,13\}$$

For each u in C_3 , we have $f^+(u) = 80$.

For $k = 5$, $f(e_1) = k$, $f(e_2) = k+1$, $f(e_3) = k+1$, $f(e_4) = k+2$, in C , the labels of splitting edges of e_1, e_2, e_3 are

$$A_1 = \{7,9,10,15,16\}, A_2 = \{2,5,8,11,14,17\} \text{ and } A_3 = \{1,3,4,6,12,13,18\}$$

For each u in C_3 , we have $f^+(u) = 114$.

When n is even, we observe that if we define $f_3 : C_4 \rightarrow N$ by $f_3(e) = 3$ for all e in C_4 . The graph $SPE(C_4, f)$ has 12 edges. It satisfies the necessary supermagic condition: $q(q+1) \equiv 0 \pmod{p}$. It is possible for us to divide $\{1,2,\dots,12\}$ into 4 group of triples $\{x_1, x_2, x_3\} : \{1,7,11\}, \{2,8,9\}$ with sum 19 and $\{3,5,12\}, \{4,6,10\}$ with sum 20. So if we label the splitting edges of (v_1, v_2) , (v_2, v_3) , (v_3, v_4) and (v_4, v_1) consecutively by $\{1,7,11\}$, $\{3,5,12\}$, $\{2,8,9\}$, $\{4,6,10\}$. Then we see that $SPE(C_4, f)$ is supermagic.

The result can be generalized to the following

Theorem 16. $SPE(C_{2n}, f)$ is supermagic if for any $t \geq 2$ $f(e_i) = t$ for all e_i in $E(C_{2n})$.

Proof. (1) When $t = 2m$ by Theorem 1, we conclude that $SPE(C_{2n}, f)$ is supermagic,

(2) Assume $t = 2m+1$, in order to prove that $SPE(C_{2n}, f)$ is supermagic_it suffices to show that it is true for $t = 3$.

Since $f(e) = 3$, we have $q = \sum f(e) = 6n$, we want to partition the set $\{1,2,\dots,6n\}$ into the following sets:

$$A(k) = \{k, 4n+k, 4n-2(k-1)\}, k = 1, 2, \dots, n,$$

$$B(k) = \{n+k, 4n-(2k-1), 5n+1\}, k = 1, 2, \dots, n,$$

The sum of numbers in $A(k)$ is $\sigma_1 = 8n + 2$, and the sum of numbers in $B(k)$ is $\sigma_2 = 10n + 1$.

Assume $E(C_{2n}) = \{e_1, e_2, \dots, e_{2n}\}$ we label the splitting edges of e_{2i} by $A(i)$ and splitting edges of e_{2i+1} by $B(i)$ $i = 1, 2, \dots, n$, then

$$f^+(u) = \sigma_1 + \sigma_2 = 3(6n + 1).$$

Therefore $SPE(C_{2n}, f)$ is supermagic. \square

Example 11. For $SPE(C_{10}, f)$ with $f(e) = 3$ for all e in C_{10} , we have $q = 30$. We partition $\{1, 2, \dots, 30\}$ into the following 3-elements subsets

$$A(1) = \{1, 20_{-21}\} \quad A(2) = \{2, 18_{-22}\} \quad A(3) = \{3, 16_{-23}\}$$

$$A(4) = \{4, 14_{-24}\} \quad A(5) = \{5, 12_{-25}\};$$

$$B(1) = \{6, 19_{-26}\} \quad B(2) = \{7, 17_{-27}\} \quad B(3) = \{8, 15_{-28}\}$$

$$B(4) = \{9, 13_{-29}\} \quad B(5) = \{10, 11_{-30}\}.$$

If we label $E(SPE(C_{10}, f))$ according to the method of the method proof of Theorem 16, then we see that $\sigma_1 = 42$ $\sigma_2 = 51$ $f^+(u) = \sigma_1 + \sigma_2 = 114$.

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