On Vertex-graceful (p,p+1)-Graphs

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ABSTRACT

A graph G with p vertices and q edges is said to be **vertex-graceful** if there exists a labeling f: $V(G)\to\{1,2,\ldots,p\}$ such that the induced labeling $f^+: E(G)\to Z_q$ defined by $f^+((u,v)) = f(u)+f(v) \pmod q$ is a bijection. The theory of vertex-graceful graphs can be viewed as the dual theory of the theory of edge-graceful graphs which was introduced by Lo [14]. The class of vertex-graceful graphs properly contain super edge-magic graphs which were introduced by Enomoto, Ringel et al [3]. In this paper we study (p,p+1)-graphs which are vertex-graceful and strong vertex-graceful

Keywords: graph labeling, graceful, edge-graceful, vertex-graceful, total edge-magic, super edge-magic.

M.R. Classification: 05C78

1.Introduction. All graphs in this paper are finite connected simple graphs with no loops or multiple edges. In this paper we introduce and study a new graph labeling problem.

A graph G with p vertices and q edge is **graceful** if there is an injective mapping f: $V\to\{0,1,\ldots,q\}$ such that $f^*: E(G)\to\{1,2,\ldots,q\}$ defined by $f^*(e)=|f(u)-f(v)|$ where e=(u,v), is surjective.

Graceful graph labelings were first introduced by Alex Rosa [15] in 1967 as means of attacking the problem of cyclically decomposing the complete graph into other graphs. A well-known conjecture made by Ringel and Kotzig stating that all trees are graceful remains unsolved. Since Rosa's original article, an enormous body of literature has grown around the subject (see [8,24]).

Another dual concept of graceful labeling on graph theory is edge-graceful labeling, which was introduced by S.P. Lo [14] in 1985. G is said to be **edge-graceful** if the edges are labeled by 1,2,3...q so that the vertex sums are distinct, mod p.

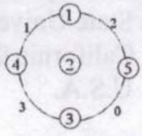
We want to introduce a new labeling problem which can be viewed as the dual of Lo's edge-graceful labeling.

Definition 1.1. A graph G is said to be **vertex-graceful** if there exists a labeling f: $V(G)\to\{1,2,\ldots,p\}$ such that the induced labeling $f^+: E(G)\to Z_q$ defined by $f^+((u,v)) = f(u) + f(v) \pmod q$ is a bijection.

We denote $f^+(\neg mod)$ the mapping $f^+(\neg mod): E(G) \to N$ defined by

$$f^+(\neg mod)((u,v)) = f(u) + f(v)$$

<u>**Example 1**</u>. The following graph $C_4 \cup K_1$ of order 5 is vertex-graceful.



$C_4 \cup K_1$ vertex-graceful

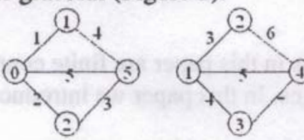
Figure 1.

The complete graph K_n is vertex-graceful if and only if $n \le 3$.

We denote the set of natural numbers by N. A subset S of N is called ***consecutive*** if S consists of consecutive integers. A mapping f: $X \to N$ from any set X to N is said to be ***consecutive*** if f (X) is consecutive.

<u>**Definition 1.2.**</u> A vertex-graceful labeling f is said to be ***strong*** if $f^+(\neg mod)$ is consecutive. If G admits a strong vertex-graceful labeling we say it is **strong vertex-graceful.**

<u>**Example 2**</u>. The graph G of order 4 and size 5 is graceful and its vertex-graceful labeling is strong vertex-graceful (Figure 2).



G is graceful G is strong vertex-graceful

Figure 2.

<u>**Example 3**</u>. The friendship graph with 7 vertices and 9 edges is not graceful. However, it is vertex-graceful. We list here two vertex-graceful labelings f,g such that g is strong but f is not (Figure 3).

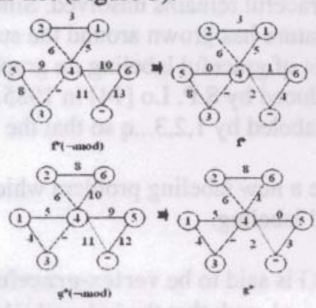


Figure 3.

The investigation of vertex-graceful graphs is initiated in [10].The class of vertex-graceful graphs properly contain super edge-magic graphs which were introduced by Enomoto, Ringel et al [3]. A (p,q)-graph G= (V, E) with p vertices and q edges is called ***total edge magic*** if there is a bijective function f : V∪ E–>{1,2,...,p+q} such that for any (u,v) in E we have a constant s with f(u) + f((u,v)) + f(v)=s. The study of total edge-magic graphs is original initiated by Kotzig and Rosa [8, 9]. They called the total edge magic graph as magic graph. Recently, Enomoto et al [3] called a total edge-magic graph as ***super edge-magic*** if f(V(G))= {1, 2,..., p}.

Chen [2] showed that a graph G is super edge-magic if and only if there exists a vertex labeling f such that the two sets f(V(G)) and {f(u)+f(v): (u,v) ∈ E(G)} are both consecutive. Independently, Figueroa-Centeno et al [5,6] showed that if f:V(G)→{1,2,…,p} is a bijection of a (p,q)-graph G with S= {f(u)+f(v): uvε E} is consecutive and s=min(S), then f can be extended to a super edge-magic labeling of G by defined f(uv)=p+q+s-f(u)-f(v) for all edge uv of E(G).

We observe that strong vertex-graceful graphs are super edge-magic. Graphs that are super edge-magic were studied in [11,13. They also related with the graphs investigated in [1]. In this paper we study (p,p+1)-graphs which are vertex-graceful and strong vertex-graceful

2. Vertex-graceful (p,p+1)-graphs of small orders and their amalgamations.

Theorem 2.1. Among five (5,6)-graphs as shown in Figure 4 only three of them are vertex-graceful.

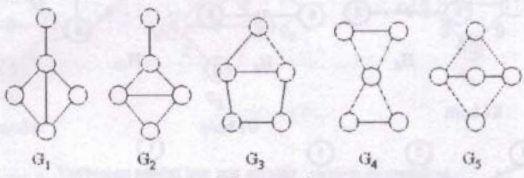


Figure 4

Proof. The graphs G_1,G_2,G_3 have the property that all vertex-graceful labelings are strong (see Figure 5).

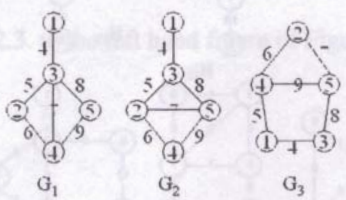


Figure 5.

Theorem 2.2. Among eighteen (6,7)-graphs as shown in Figure 6 only four of them are non vertex-graceful.
Proof. The following four (6,7)-graphs are not vertex-graceful. (Figure 6)

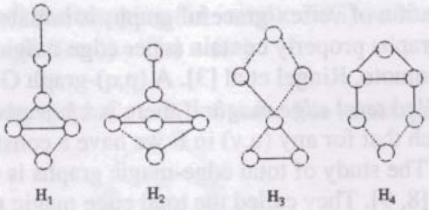


Figure 6.

The following fourteen (6,7)-graphs are vertex-graceful. (Figure 7). We note that except H_{18} is vertex-graceful but not strong vertex-graceful, all the others are strong vertex-graceful.

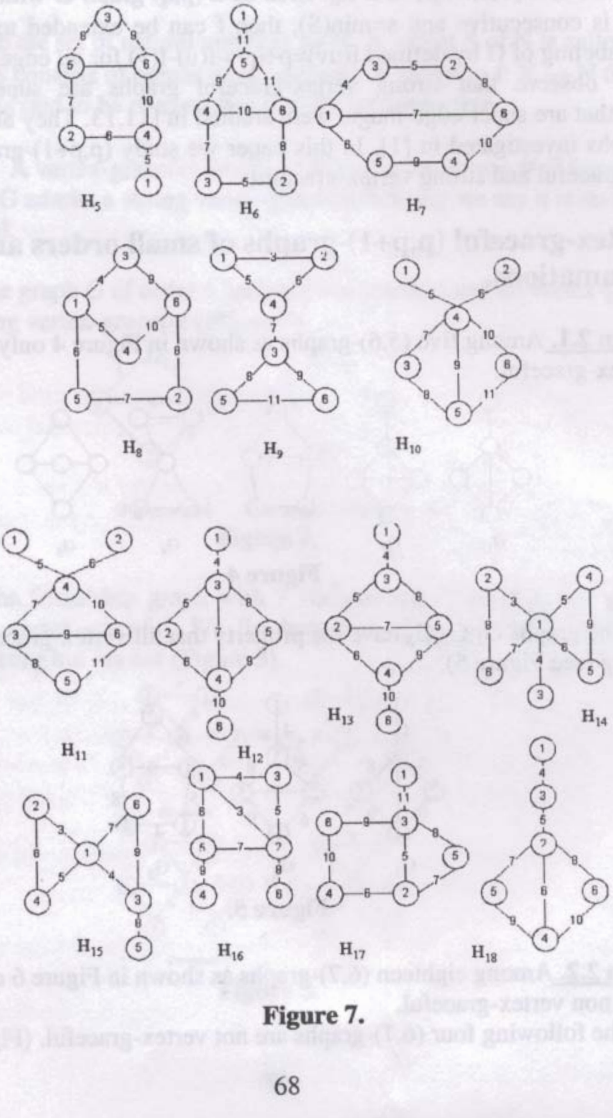


Figure 7.

Let G and H be two graphs with $u \in G$ and $v \in H$. The amalgamation of (G,u) with (H,v) is the graph obtained by forming the disjoint union of G and H and then identifying u and v. We will use Amal(G,H;(u,v)) to denote the amalgamation of (G,u) and (H,v). Let ST(m) be a star with $V(ST(m)) = \{u_0, x_1, \ldots, x_m\}$ and u_0 is the center of the star. In [10] Lee showed a construction of strong vertex-graceful graphs by amalgamation.

Theorem 2.3. If G is a strong vertex graceful (p,p+1)-graph with labeling f and if $\max f^+(E(G)) = h$ and $f(v) = h-p+1$ then the amalgamation Amal(G, ST(k); (v, u_0)) is strong vertex-graceful for all $k \geq 1$.

Theorem 2.4. If $H_1(k) = Amal(H_1, ST(k);(c_{1,1}, u_0))$ then it is vertex-graceful for all even k but not vertex-graceful for k =3,5,7.

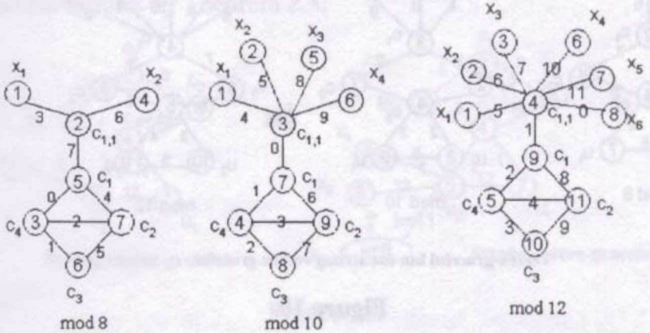
Proof. Assume k =2t. The order of $H_1(k)$ is 2t+5. Assume $V(H_1(2t)) = \{x_1, \ldots, x_{2t}, c_{1,1}, c_1, c_2, c_3, c_4\}$.

Define $f: V(H_1(k)) \rightarrow \{1,2,\ldots,2t+5\}$ as follows:

$f(x_i) = i$ for $i = 1,2,\ldots t$

$f(x_i) = i+2$ for $i = t+1, t+2, \ldots 2t$

$f(c_{1,1}) = t+1$, $f(c_1) = 2t+3$, $f(c_2) = 2t+5$, $f(c_3) = 2t+4$, $f(c_4) = t+2$.

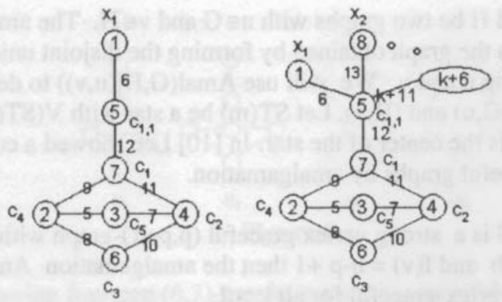


Vertex-graceful but not strong vertex-graceful

Figure 8.

Theorem 2.4. If $H_2(k) = Amal(H_2, ST(k);(c_{1,1}, u_0))$ then it is strong vertex-graceful for all $k \geq 1$.

Proof. Apply Theorem 2.3. to the left hand figure of Figure 9 we obtain the result directly.

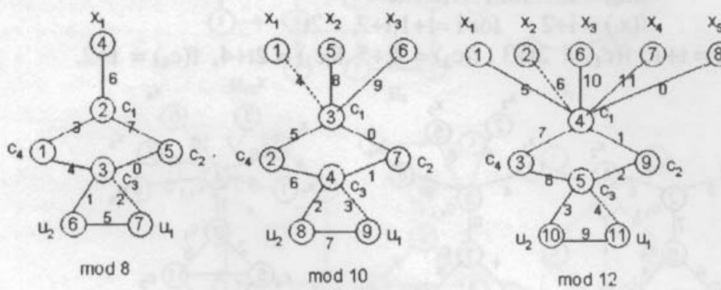


Strong vertex-graceful   Strong vertex-graceful

Figure 9.

Theorem 2.5. If $H_{3,1}(k) = Amal(H_3, ST(k);(c_1 , u_0))$ then it is vertex-graceful for all odd integer $k \geq 1$.

Proof.

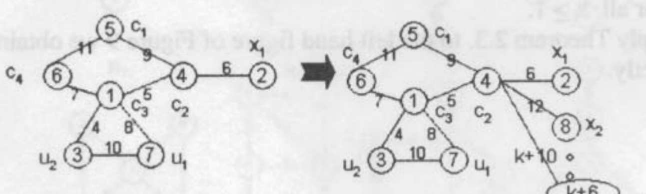


Vertex-graceful but not strong vertex-graceful

Figure 10.

Theorem 2.6. If $H_{3,2}(k) = Amal(H_3, ST(k);(c_2 , u_0))$ then it is strong vertex-graceful for all integer $k \geq 1$.

Proof. Apply Theorem 2.3. to the left hand figure of Figure 11 we obtain the result directly.



Strong vertex-graceful   Strong vertex-graceful

Figure 11.

Theorem 2.7. If $H_{3,3}(k) = Amal(H_3, ST(k);(c_3, u_0))$ then it is strong vertex-graceful for all integer $k \geq 1$.

Proof. Apply Theorem 2.3. to the left hand figure of Figure 12 we obtain the result directly.

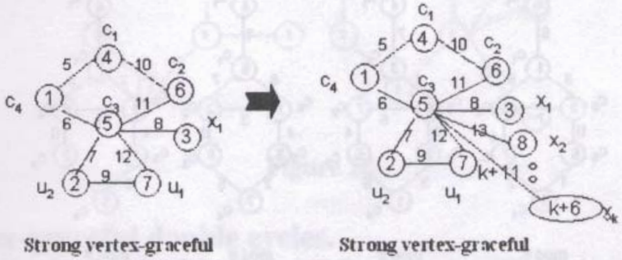


Figure 12.

Theorem 2.8. If $H_{3,3}(k) = Amal(H_3, ST(k);(u_2, u_0))$ then it is strong vertex-graceful for all integer $k \geq 1$.

Proof. Left figure shows that $H_{3,3}(1)$ is strong vertex-graceful. $H_{3,3}(k)$ is strong vertex-graceful follows by Theorem 2.3.

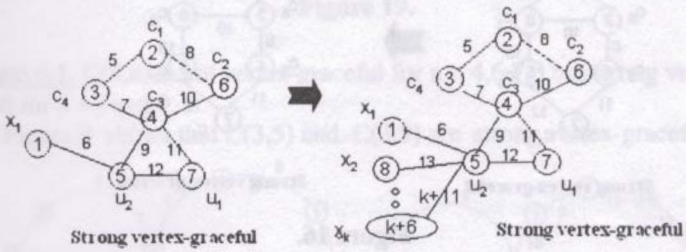


Figure 13.

Similarly we have

Theorem 2.9. The graph $Amal(H_3, ST(k);(c_{1,1}, u_0))$ is strong vertex-graceful for all $k \geq 1$.

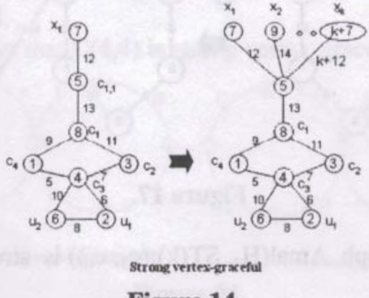


Figure 14.

Theorem 2.10. The graph Amal(H_4, ST(k);(c_1, u_0)) is vertex-graceful for k =1 and strong vertex-graceful for all $k \geq 2$.

Proof.

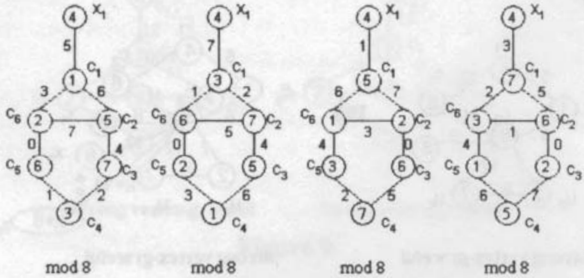


Figure 15.

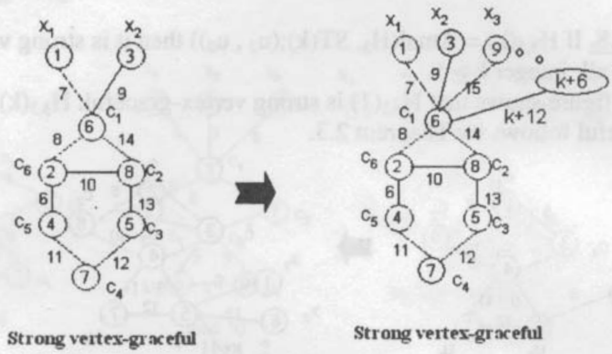


Figure 16.

Theorem 2.11. The graph Amal(H_4, ST(k);(c_2, u_0)) is strong vertex-graceful for all $k \geq 1$.

Proof.

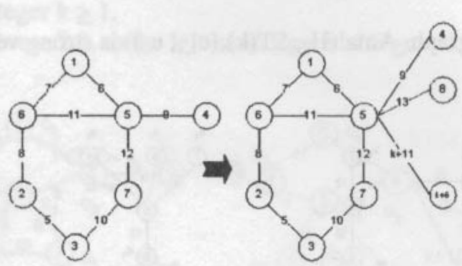


Figure 17.

Theorem 2.12. The graph Amal(H_4, ST(k);(c_3, u_0)) is strong vertex-graceful for all $k \geq 1$.

Proof.

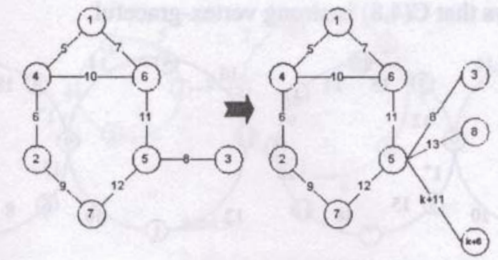
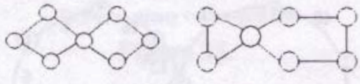


Figure 18.

3. Vertex-graceful double cycles.

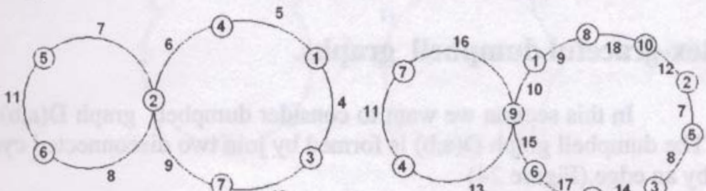
The following graphs C(4,4) and C(3,5) are the Amal(C_4,C_4,(u,u)) and Amal(C_3,C_5,(u,v)) respectively (Figure 8). We will denote Amal(C_m, C_n) by C(m,n) and called it **double cycle.**



C(4,4) C(3,5)

Figure 19.

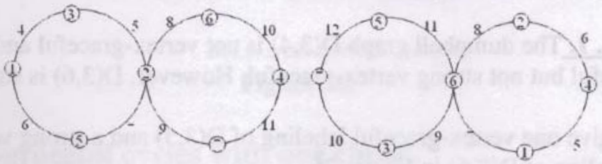
<u>**Theorem 3.1.**</u> C(3,n) is not vertex-graceful for n = 4,6,7,8 but strong vertex-graceful for n =5 and 9.
Proof. Figure 9 shows that C(3,5) and C(3,9) are strong vertex-graceful.



C(3,5) is strong vertex-graceful C(3,9) is strong vertex-graceful

Figure 20.

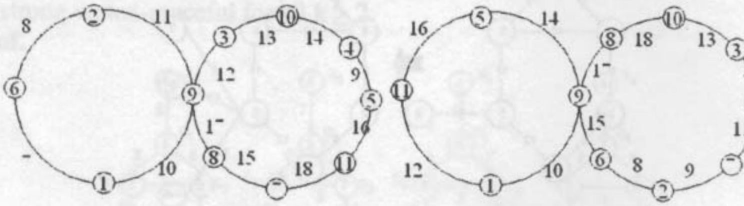
<u>**Theorem 3.2.**</u> C(4,n) is not vertex-graceful for n = 5,6,7,9 but strong vertex-graceful for n =4 and 8.
Proof. Figure 21 shows that C(4,4) is strong vertex-graceful.



C(4,4) is strong vertex-graceful C(4,4) is strong vertex-graceful

Figure 21.

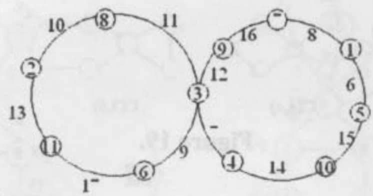
Figure 22 shows that C(4,8) is strong vertex-graceful.



C(4,8) is strong vertex-graceful C(4,8) is strong vertex-graceful

Figure 22.

__Theorem 3.3.__ C(5,n) is not vertex-graceful for n = 5,6,8,9 but strong vertex-graceful for n = 7.
Proof.



C(5,7) is strong vertex-graceful

Figure 23.

4.Vertex-graceful dumpbell graphs.

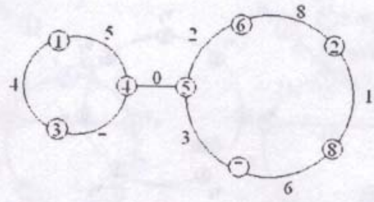
In this section we want to consider dumpbell graph D(a,b) where a+b=p. The dumpbell graph D(a,b) is formed by join two disconnected cycles C_a and C_b by an edge.(Figure 24)

D(3,4)

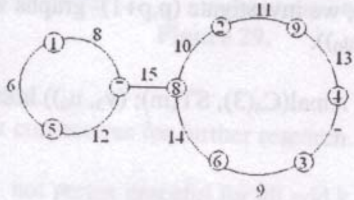
Figure 24.

__Theorem. 4. 1.__ The dumpbell graph D(3,4) is not vertex-graceful and D(3,5) is vertex-graceful but not strong vertex-graceful. However, D(3,6) is strong vertex-graceful.

Proof. We give one vertex-graceful labeling of D(3,5) and a strong vertex-graceful labeling of D(3,6) in Figure 25.



mod 9

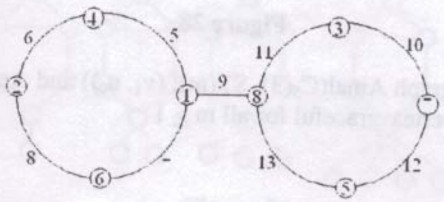


D(3.6) is strong vertex-graceful

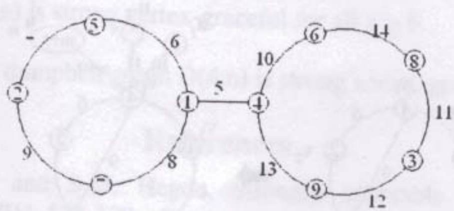
Figure 25.

<u>**Theorem 4.3.**</u> The dumpbell graph D(4,n) is strong vertex-graceful for n = 4,5 but vertex-graceful for n =6.

Proof.



D(4.4) is strong vertex-graceful



D(4.5) is strong vertex-graceful

Figure 26.

5. Vertex-graceful cycles with one chord.

We consider the (p,p+1) graphs : cycles with a chord in this section. Assume the vertices of cycle are$\{v_1,v_2,\ldots,v_n\}$ and the chord connect vertex v_1 with v_r, we denote this graph by $C_n(r)$.

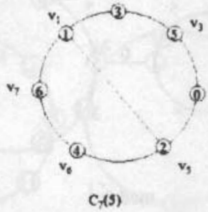


Figure 27.

In this section, we investigate (p,p+1)- graphs which are of the form Amal(C_n(3), ST(m); (v_2, u_0)).

Theorem 5.1. The graph Amal(C_4(3), ST(m); (v_3, u_0)) is strong vertex-graceful for all $m \geq 1$

Proof.

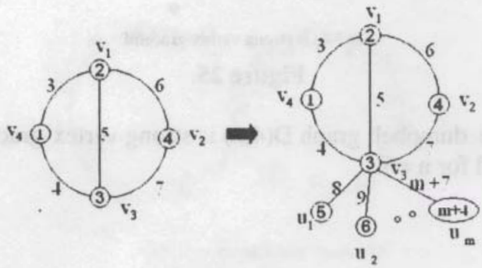
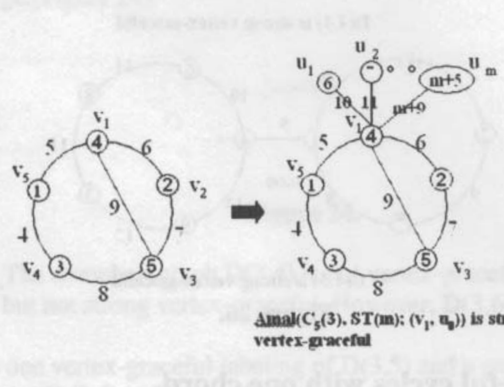


Figure 28.

Theorem 5.2. The graph Amal(C_5(3), ST(m); (v_1, u_0)) and Amal(C_5(3), ST(m); (v_2, u_0)) are strong vertex-graceful for all $m \geq 1$

Proof.



Amal(C_5(3), ST(m); (v_1, u_0)) is strong vertex-graceful

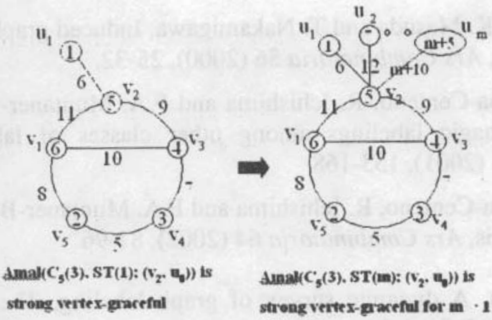


Figure 29.

6. Some Conjectures.

We propose here six conjectures for further research.

<u>**Conjecture 1**</u>. $H_1(k)$ is not vertex graceful for all odd $k \geq 9$.

<u>**Conjecture 2**</u>. $H_2(k)$ is not vertex graceful for all $k \geq 6$.

<u>**Conjecture 3**</u>. $H_{3,1}(k) = \text{Amal}(H_3, ST(k);(c_1, u_0))$ is not vertex-graceful for all even integer $k \geq 1$.

<u>**Conjecture 4**</u>. $H_4(k)$ is not vertex graceful for all $k \geq 6$.

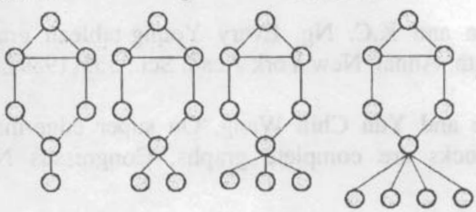


Figure 31.

<u>**Conjecture 5.**</u> D(3,n) is strong vertex-graceful for all $n \geq 6$.

<u>**Conjecture 6.**</u> The dumpbell graph D(4,n) is strong vertex-graceful for all $n \geq 4$.

References

[1] B.D. Acharya and S.M. Hegde, Strongly indexable graphs, Discrete Mathematics, 93(1991), 123-129.

[2] Z. Chen, On super edge-magic graphs, *The Journal of Combinatorial Mathematics and Combinatorial Computing* **38**(2001),53-64.

[3] H. Enomoto, A. S. Llato, T. Nakamigawa, A. Ringel, Super edge-magic graphs, *SUT J. Math.* **43** (2) (1998), 105-109.

[4] H. Enomoto, K. Masuda and T. Nakamigawa, Induced graph theorem on magic valuations, *Ars Combinatoria* **56** (2000), 25-32.

[5] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle , The place of super edge-magic labelings among other classes of labelings, *Discrete Mathematics* **231** (2001), 153-168

[6] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle , On super edge-magic graphs, *Ars Combinatoria* **64** (2002), 81-96.

[7] J.A. Gallian, A dynamic survey of graph labeling, *The Electronic J. of Combin*. (2001), # DS6, 1-144.

[8] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canada Math. Bull. 1*3 (1970), 451-461.

[9] A. Kotzig and A. Rosa, Magic valuations of complete graphs, Publications du Centre de Recherches Mathematiques Universite de Montreal 175 (1972)

[10] Sin-Min Lee, On Strongly Indexable Graphs and Super Vertex-graceful Graphs, manuscript.

[11] Sin- Min Lee and Alexander Nien-Tsu Lee, On super edge-magic graphs with many odd cycles, Congressus Numerantium 163 (2003) , 65-80.

[12] Sin-Min Lee and K.C. Ng, Every Young-tableau graph is d-graceful, Combinatorial Math, Annal. New York Acad. Sci., 555(1989), 296-302.

[13] Sin-Min Lee and Yun Chin Wang, On super edge-magicness of chain graphs whose blocks are complete graphs, Congressus Numerantium 162 (2003), 147-160.

[14] S.P. Lo, On edge-graceful labelings of graphs, *Congressus Numerantium*, **50** (1985)., 231-241.

[15] A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs (International Symp. Rome, July, 1966), Gordon and Breach, N.Y. and Paris,1967, 349-355.