On The Integer -Magic Spectra of Two-vertex sum of Paths

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Abstract: For k>0, we call a graph G=(V,E) as **Z_k-magic** if there exists a labeling $l:E(G)\rightarrow Z_k^*$ such that the induced vertex set labeling $l^+: V(G)\rightarrow Z_k$ defined by $l^+(v)=\Sigma$ {l(u,v) : (u,v) in E(G)}is a constant map. We denote the set of all k such that G is k-magic by IM(G). We call this set as the **integer-magic spectrum** of G. We investigate the integer-magic spectra for graphs which are two-vertex sum of paths.

1. **Introduction.** For any abelian group A, written additively we denote A∗ = A−{0}. Any mapping l: E(G) → A∗ is called a labeling. Given a labeling on edge set of G we can induced a vertex set labeling l^+: V(G) → A as follows:

$$l^+(v)=\Sigma \{l(u,v) : (u,v) \text{ in } E(G)\}$$

A graph G is known as A-magic if there is a labeling l: E(G) → A∗ such that for each vertex v, the sum of the labels of the edges incident with v are all equal to the same constant; i.e., $l^+(v) = c$ for some fixed c in **A**. We will called <G,l> a A-magic graph. In general, a graph G may admit more than one A-magic labelings.

We denote the class of all graphs (either simple or multiple graphs) by **Gph**. The class of all abelian groups by **Ab**. For each A in **Ab** we denote the class of all A-magic graphs by $_A$**MGp.**

When A = Z, the Z-magic graphs were considered in Stanley[26,27]; he pointed out that the theory of magic labelings can be put into the more general context of linear homogeneous diophantine equations [26].

When the group is Z_k, we shall refer to the Z_k –magic graph as *k-magic*. Graphs which are k-magic had been studied in [10, 12,13,14,15,18,19].

The original concept of A-magic graph is due to J. Sedlacek [23,24], who defined it to be a graph with real-valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

Doob [1,2,3] also considered A-magic graphs where A is an abelian group. Given the graph G , the problem of deciding whether G admits a magic labeling is equivalent to the problem of deciding whether a set of linear

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homogeneous Diophantine equation has a solution. At present, given an abelian group, no general efficient algorithm is known for finding magic labelings for general graphs.

In this paper we use **N** to denote the set $\{1,2,3,\ldots\}$ and for each $k>0$, we write the set $\{kx: x \text{ in } N\}$ by **kN** and $\{k+x: x \text{ in } N\}$ by **k+N**. We will define the graph G with a magic labeling $l: E(G) \rightarrow N$ as **N-magic**. It is well-known that a graph G is N-magic if and only if each edge of G is contained in a 1-factor (a perfect matching) or a $\{1,2\}$-factor (see [9,22,30,31]). Reader refer to [4,5,6,7,8,10,12,24,25,28,29] for N-magic graphs. The Z-magic is weaker than N-magic. Figure 1 shows a graph which is Z-magic but not N-magic.

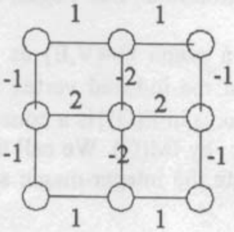
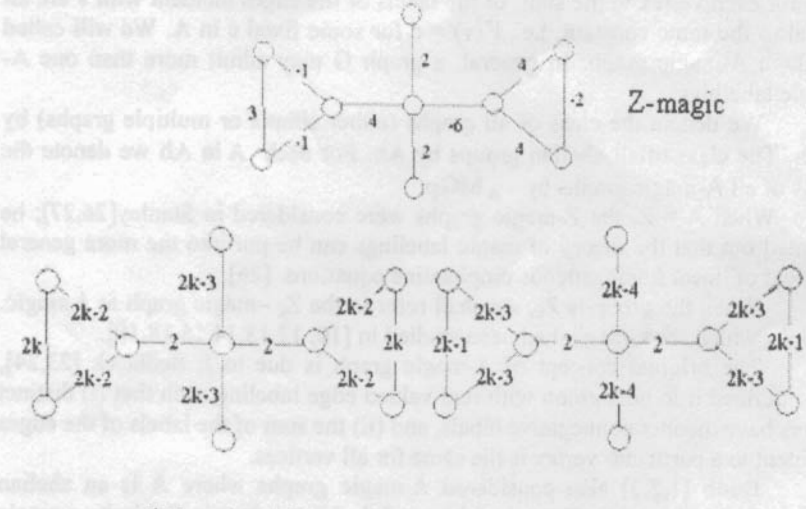


Figure 1.

For simplicity, we will consider Z-magic as 1-magic. Given a graph G, we denote the set of all $k>0$ such that G is k-magic by IM(G). We call this set as **integer-magic spectrum** of G. We investigate these sets for general graphs in [12,16]. Figure 2 shows a graph G whose $IM(G)= N-\{2,3,4\}$.



2k+1-magic , k≥ 2 2k-magic , k ≥ 3.

Figure 2.

A special type of amalgamation of graphs called the **two-vertex sum** was considered in [11] by the first author.

Let $\Omega = \{(G_i, \{u_i, v_i\}) : i \in I\}$ be a class of graphs with two distinguished vertices u_i and v_i. The two vertex-sum of Ω is the disjoint union of G_i by identification all u_i and identification all v_i, under the following edge identification:

(1) for those vertices (u_i, v_i) which are adjacent we identify the edges into one.
(2) for those vertices (u_i, v_i) with distance $d((u_i, v_i) > 1$, we keep the paths between them.

We will denote this resulting graph by $\Sigma(\Omega)$.

Example 1. Figure 3 shows a two-vertex sum of three cycles.

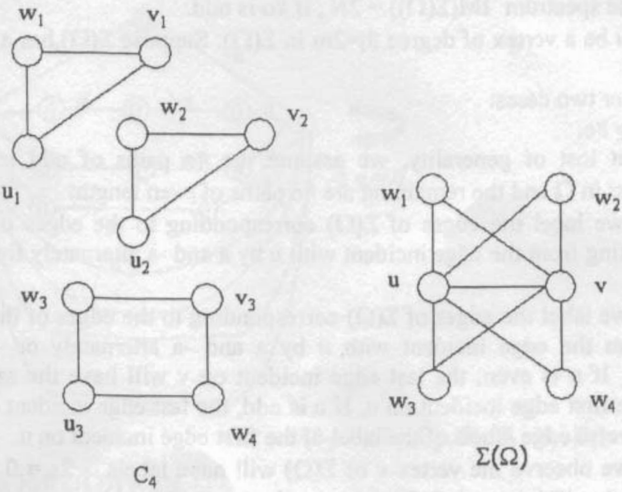


Figure 3.

In this paper, we will consider the integer-magic spectra of graphs that are constructed from the two-vertex sum of paths. In particular, we completely determine the integer-magic spectra of the generalized theta graphs

2.Generalized ThetaGraphs.

If $\Omega = \{(P_i, \{u_i, v_i\}) : i \in I\}$ is a collection of paths with two ends u_i and v_i then $\Sigma(\Omega)$ is the generalized theta graph. If $|I|=2$, then $\Sigma(\Omega)$ is isomoprhic to a cycle. Thus $IM(\Sigma(\Omega)) = N$. So we may assume $|I| > 2$.

By our definition of two vertex sum if there are more than two P_2 in Ω, all these P_2 will identified in $\Sigma(\Omega)$. Thus we may assume in Ω, there is at most one P_2.

We consider two cases.

(a)|I| is even.

Suppose $\Omega = \{(P_i, \{u_i, v_i\}) : i \in I\}$and $|I|$ is even and greater than 2.

Let #o = total number of n_i which are odd and #e = total number of n_i which are even.

Theorem 2.1. Given $\Omega = \{(P_i, \{u_i, v_i\}) : i \in I\}$ with $|I|$ even, we have the integer magic spectrum $IM(\Sigma(\Omega)) = N$, if #o is even.

Proof. The graph $\Sigma(\Omega)$ is eulerian with even number of edges. Low and Lee [21] showed that if a graph G is eulerian of even size then it is A-magic for any abelian group A. In particular, $\Sigma(\Omega)$ is k-magic for all k.

Thus, we have $IM(\Sigma(\Omega)) = N$.□

Example 2. Figure 4 illustrates the result for $\theta(P_2, P_4, P_5, P_5)$**Figure 4**

Theorem 2. 2. Given $\Omega = \{(P_i, \{u_i, v_i\}) : i \in I\}$ with $|I|$ even, we have the integer magic spectrum $IM(\Sigma(\Omega)) = 2N$, if #o is odd.

Proof. Let *u* be a vertex of degree $|I|=2m$ in $\Sigma(\Omega)$. Suppose $\Sigma(\Omega)$ has a k-magic labeling.

Consider two cases:

Case 1. #o ≥ #e.

Without lost of generality, we assume the #o paths of odd length are arranged first in Ω and the remaining are #e paths of even length.

Then we label the edges of $\Sigma(\Omega)$ corresponding to the edges of the 2m paths P starting from the edge incident with *u* by a and -a alternately from top to bottom.

Next, we label the edges of $\Sigma(\Omega)$ corresponding to the edges of the path P_n starting from the edge incident with *u* by a and -a alternately or –a and a alternately. If n is even, the last edge incident on v will have the same edge labels as the first edge incident on u. If n is odd, the last edge incident on v will have the inverse edge labels of the label of the first edge incident on u.

Then we observe the vertex v of $\Sigma(\Omega)$ will have labels $-2a \equiv 0 \pmod{k}$. This shows that if the graph $\Sigma(\Omega)$ is k –magic then k should be an even number. (Figure 5 illustrates the case for $\Sigma(P_2, P_2, P_3, P_4, P_5, P_5)$

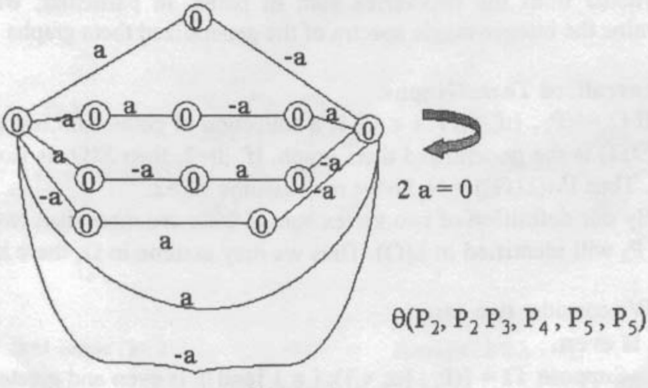


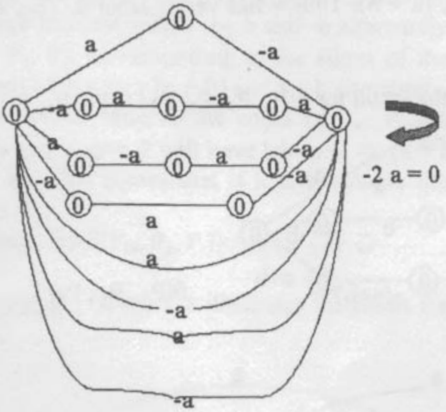
Figure 5.

Case 2. #o < #e.

Without lost of generality, we assume the #o paths of odd length are arranged first in Ω and the remaining are #e paths of even length.

Then we label the edges of $\Sigma(\Omega)$ corresponding to the edges of the 2m paths P starting from the edge incident with u by a and -a alternately from top to bottom.

Next, we label the edges of $\Sigma(\Omega)$ corresponding to the edges of the path P_n starting from the edge incident with u by a and -a alternately or –a and a alternately. If n is even, the last edge incident on v will have the same edge labels as the first edge incident on u. If n is odd, the last edge incident on v will have the inverse edge labels of the label of the first edge incident on u.
Then we observe the vertex v of $\Sigma(\Omega)$ will have labels $-2a = 0 (\mathrm{mod}\ k)$. This shows that if the graph $\Sigma(\Omega)$ is k–magic then k should be an even number. (Figure 6 illustrates the case for $\Sigma(P_2, P_2, P_2, P_2, P_3, P_4, P_5, P_5)$



$\theta(P_2, P_2, P_2, P_2, P_3, P_4, P_5, P_5)$ **Figure 6.**

Hence, $IM(\Sigma(\Omega)) = 2N$. □

(b)|I| is odd.

Assume $|I| = 3$. Now, let P_m, P_n, and P_t be three paths, where m, n, and t are at least 2 and m, n, and t are the number of vertices of each of the paths P_m, P_n, and P_t correspondingly. We denote $\Sigma(\Omega)$ by $\theta(P_m, P_n, P_t)$.

Theorem 2. 3. $IM(\theta(P_m, P_n, P_t)) = N - \{2\}$, where m, n, and t are even or m, n, t are odd and greater or equal 3.

Proof. Since the graph $\theta(P_m, P_n, P_t)$ has a vertex of degree 2 and a vertex of degree 3, therefore it is not 2-magic. Now we want to show that it is k-magic for any $k \neq 2$.

Case 1. Assume m, n, and t are even and greater than 3.

We want to show that it admits a k-magic labeling with sum 0. Let u be a vertex of degree 3 of $\theta(P_m, P_n, P_t)$. We label the edges of $\theta(P_m, P_n, P_t)$ corresponding to the edges of the path P_m starting from the edge incident with u by a and -a alternately. Next, we label the edges of $\theta(P_m, P_n, P_t)$ corresponding to the edges of the path P_n starting from the edge incident with u by b and -b alternately.

We observe the edges of P_m, P_n and P_t which connect to the end vertex v of degree 3 will have labels a, b, -(a + b). Thus v has vertex label 0. This shows that $\theta(P_m, P_n, P_t)$ is k -magic.

Case 2. Assume *m*, *n*, and *t* are odd and greater than 2.

We want to show that it admits a k-magic labeling with sum 0. Let *u* be a vertex of degree 3 of $\theta(P_m, P_n, P_t)$. We label the edges of $\theta(P_m, P_n, P_t)$ corresponding to the edges of the path P_m starting from the edge incident with *u* by a and -a alternately. Next, we label the edges of $\theta(P_m, P_n, P_t)$ corresponding to the edges of the path P_n starting from the edge incident with *u* by b and -b alternately.

We observe the edges of P_m, P_n and P_t which connect to the end vertex v of degree 3 will have labels -a, -b, (a + b). Thus v has vertex label 0. This shows that $\theta(P_m, P_n, P_t)$ is k -magic.□.

Example 3. Figure 7 illustrates the result for $\theta(P_3, P_5, P_5)$ and $\theta(P_2, P_4, P_6)$

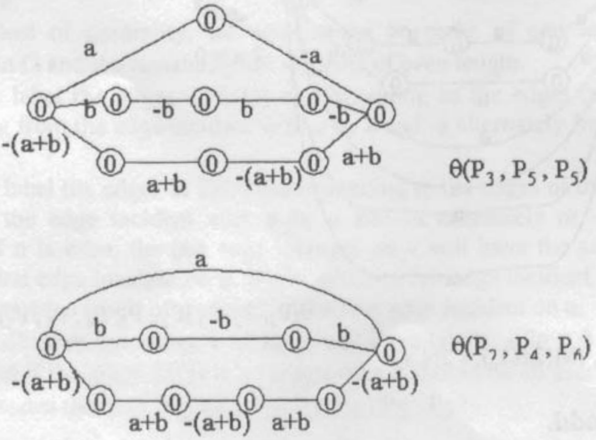


Figure 7.

Theorem 2. 4.. $IM(\theta(P_m, P_n, P_t)) = 2N - \{2\}$, where exactly one of the numbers *m*, *n*, and *t* is odd or where *m*, *n*, and *t* are greater than 2 and exactly two of the numbers *m*, *n*, and *t* are odd.

Proof. Since the graph $\theta(P_m, P_n, P_t)$ has a vertex of degree 2 and a vertex of degree 3, therefore it is not 2-magic. Now we want to show that it is k-magic for any $k \neq 2$.

Case 1. Assume one of the numbers *m*, *n*, and *t* is odd. Without loss of generality let *t* be odd. Let *u* be a vertex of degree 3 of $\theta(P_m, P_n, P_t)$. Consider an k-magic labeling of $\theta(P_m, P_n, P_t)$. Then we label the edges of $\theta(P_m, P_n, P_t)$ corresponding to the edges of the path P_m starting from the edge incident with *u* by a and -a alternately. Next, we label the edges of $\theta(P_m, P_n, P_t)$ corresponding to the edges of the path P_n starting from the edge incident with *v* by b and -b alternately. Finally, we label the edges of $\theta(P_m, P_n, P_t)$ corresponding to the edges of the path P_t starting from the edge incident with *u* by $-(a + b)$ and $a + b$, alternately.

Then we observe the edges of P_m, P_n and P_t which connect to the end vertex v of degree 3 will have labels a, b, $(a + b)$. Thus v has vertex label $2(a + b) \equiv 0 \pmod{k}$. This shows that if it is k –magic then k should be an even number except 2.

Case 2. Let *m*, *n*, and *t* are greater than 2 and exactly two of the numbers *m*, *n*, and *t* are odd. Without loss of generality we let *n* and *t*, be odd. Let *u* be a vertex of degree 3 of $\theta(P_m, P_n, P_t)$. Consider an k-magic labeling of $\theta(P_m, P_n, P_t)$. Then we label the edges of $\theta(P_m, P_n, P_t)$ corresponding to the edges of the path P_m starting from the edge incident with *u* by a and -a alternately. Next, we label the edges of $\theta(P_m, P_n, P_t)$ corresponding to the edges of the path P_n starting from the edge incident with *v* by b and -b alternately. Finally, we label the edges of $\theta(P_m, P_n, P_t)$ corresponding to the edges of the path P_t starting from the edge incident with *u* by $-(a + b)$ and $a + b$ alternately.

Then we observe the edges of P_m, P_n, and P_t which connect to the end vertex v of degree 3 will have labels a, -b, $(a + b)$. Thus v has vertex label $2a \equiv 0 \pmod{k}$. This shows that if it is k –magic then k should be an even number except 2.

Hence, $IM(\theta(P_m, P_n, P_t)) = 2N - \{2\}$. □

Example 4. Figure 8 illustrates the result for $\theta(P_2, P_4, P_5)$ and $\theta(P_4, P_5, P_5)$

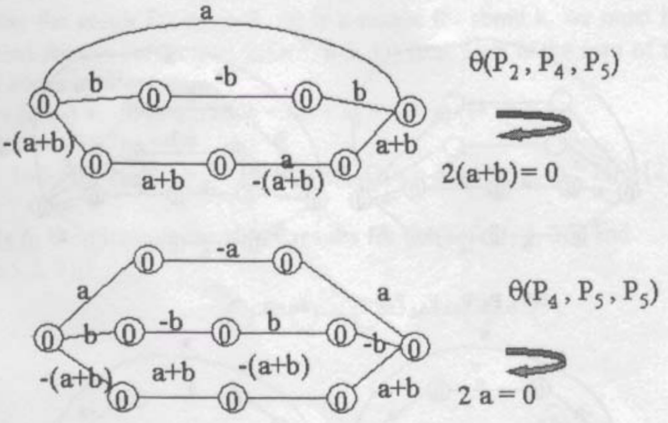


Figure 8.

We have the following surprising result.

Theorem 2. 5.. Given $\Omega = \{(P_i, \{u_i, v_i\}) : i \in I\}$ with $|I|$ odd greater than 3, we have the integer magic spectrum

(a) $IM(\Sigma(\Omega)) = N\backslash\{2\}$, if #o is even

(b) $IM(\Sigma(\Omega)) = 2N\backslash\{2\}$, if #o is odd

Proof. It is clear that $\Sigma(\Omega)$ is not 2-magic.

(a)If #o is even then #e is odd.

For any k in $N\setminus\{2\}$. We want to show that $\Sigma(\Omega)$ is k-magic.

Let u be a vertex with maximum degree in $\Sigma(\Omega)$. Consider the induced subgraph G of $\Sigma(\Omega)$ which consists of all #o paths of odd length and (#e)- 3 paths with even length.. It is obvious that it has a k-magic labeling with zero sum.

Now apply Theorem 2.3. for the remaining unlabeled subgraph of $\Sigma(\Omega)$, which is of the form $\theta(P_m, P_n, P_t)$ with three odd path length. We can have a k-magic labeling $\theta(P_m, P_n, P_t)$ with zero sum. Combine this labeling with the one in G, we can have a k-magic labeling of $\Sigma(\Omega)$.

(b) If #o is odd then #e is even.

For any k in $2N\setminus\{2\}$. We want to show that $\Sigma(\Omega)$ is k-magic.

Let u be a vertex with maximum degree in $\Sigma(\Omega)$. Consider the induced subgraph G of $\Sigma(\Omega)$ which consists of all (#o)-1 paths of odd length and (#e)- 2 path with even lengths. It is obvious that it has a k-magic labeling with zero sum. Now apply Theorem 2.4. for the remaining unlabeled subgraph of $\Sigma(\Omega)$,which is of the form $\theta(P_m, P_n, P_t)$ with two odd path length and one even path length. We can have a k-magic labeling $\theta(P_m, P_n, P_t)$ with zero sum. Combine this labeling with the one in G, we have a k-magic labeling of $\Sigma(\Omega)$.□

We illustrate the above result by the following

Example 5. We illustrate the above results for $IM(\theta(P_2,P_4,P_4,P_5,P_5))$ and $IM(\theta(P_2,P_3,P_4,P_5,P_5))$

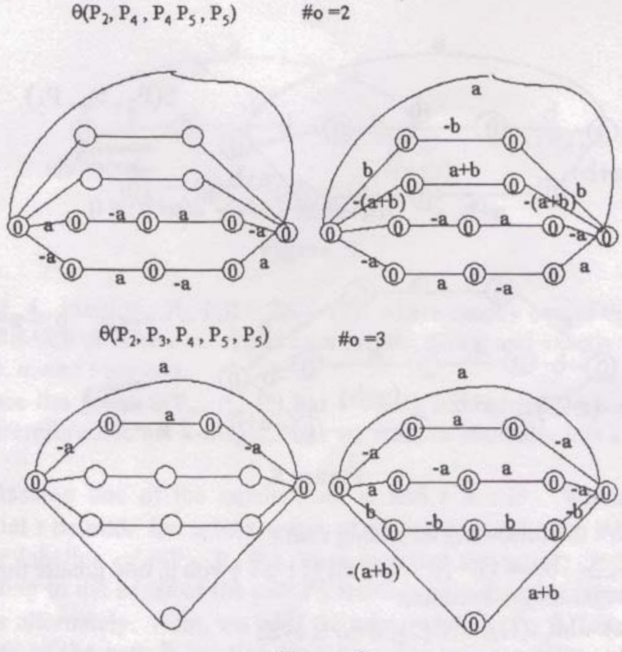


Figure 9.

3. General cases.

Let $\Omega(n, t, s) = \{(P_i, \{u_i, v_i\}) : P_i = P_n$ for each $i = 1, 2, \ldots, s$ and $d(u_i, v_i) = t\}$ is a collection of paths with two distinguish vertices u_i and v_i with distance t.

Then $\Sigma(\Omega(n, n\text{-}1, s))$ is the generalized theta graph.

Theorem 3.1 $IM(\Sigma(\Omega(n, n\text{-}r, s))) = \varnothing$ if $r > 3$.

Proof. If $r > 2$, the graph .$\Sigma(\Omega(n, n\text{-}r, s))$ has a pendant path of length 2. Hence it is not k-magic for any k. □

Theorem 3.2 If n is odd integer greater than 3, then the integer magic spectrum $IM(\Sigma(\Omega(n, n\text{-}3, s))) = (\cup\, p_i^{ai} N) \setminus \{2\}$ where $3s\text{-}2 = p_1^{ai} p_2^{a2} \ldots p_m^{am}$.

Proof. We note first $\Sigma(\Omega(n, n\text{-}3, s))$ cannot be 2-magic.

Let u and v be two non-end vertices of P_n whose distance is n-3. Suppose we have s copies of P_n and we form the two vertex sum $\Sigma(\Omega(n, n\text{-}3, s))$of $\Omega(n,n\text{-}3,s)$.

Suppose we label the edges in the first path P_n of $\Sigma(\Omega(n, n\text{-}3, s))$ by a, x_1, a-x_1, x_1, a-x_1, …, a-x_1, a.

We label the edges in the second path P_n of $\Sigma(\Omega(n, n\text{-}3, s))$ by a, x_2,a-x_2, x_2,.a-x_2, …,a-x_2, a.

For the ith path P_n, where $i < s$, we label the edges in the ith path P_n of $\Sigma(\Omega(n, n\text{-}3, s))$ by a, x_i, a-x_i, x_i, a-x_i, …,a-x_i, a.

Finally, for the sth path P_n of $\Sigma(\Omega(n, n\text{-}3, s))$ we will label the edges by a, $-(s-1)a-(x_1 + x_2 + \ldots + x_{s-1})$, $sa + (x_1 + x_2 + \ldots + x_{s-1})$, $-(s-1)a-(x_1 + x_2 + \ldots + x_{s-1})$, …, $sa + (x_1 + x_2 + \ldots + x_{s-1})$, a.

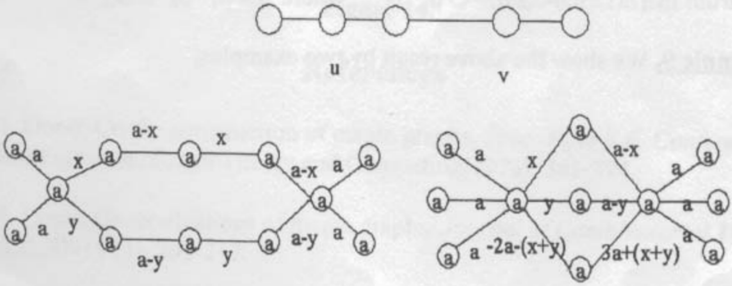
In order the graph $\Sigma(\Omega(n, n\text{-}3, s))$ is k-magic for some k, we must have the vertex label for the vertex v in $\Sigma(\Omega(n, n\text{-}3, s))$ is a. That is the sum of the edge labels of edges incident on v is

$$(a-x_1) + (a-x_2) + \ldots + (a-x_{s-1}) + sa + (x_1 + x_2 + \ldots + x_{s-1}) + sa = a$$

implying that $(3s-2)\, a = 0$.

Thus if $3s\text{-}2 = p_1^{ai} p_2^{a2} \ldots p_m^{am}$, then $IM(\Sigma(\Omega(n,n\text{-}3,s))) = (\cup\, p_i^{ai} N) \setminus \{2\}$. □

Example 6. We illustrate the above results for $IM(\Sigma(\Omega(7, 4, 2)))$ and $IM(\Sigma(\Omega(5,2, 3)))$

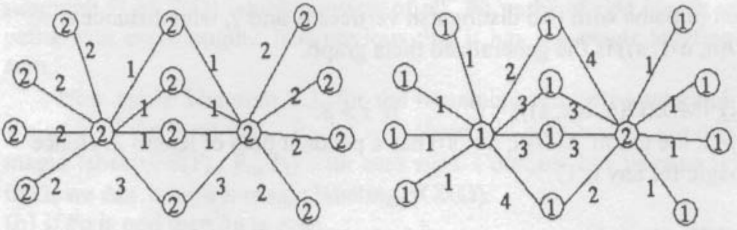


$IM(\Sigma(\Omega(7,4,2)))= 4\,N$

$IM(\Sigma(\Omega(5,2,3)))= 7\,N$

Figure 10.

Example 7. The integer magic spectrum for $IM(\Sigma(\Omega(5,2,4)))$ is $2N \cup 5N \setminus \{2\}$.



$\Sigma(\Omega(5,2,4))$ is 4-magic $\Sigma(\Omega(5,2,4))$ is 5-magic

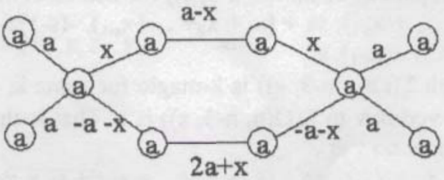
Figure 11.

Similarly, we have the following results.

Theorem 3. 3 If n is even integer greater than 3, then we have the integer magic spectrum $IM(\Sigma(\Omega(n,n-3,s))) = N \setminus \{2\}$ for all $s \geq 2$.

Since he proof is similar to Theorem 3.2, hence we skip the proof .

Example 8. Figure 12 illustrates the above result.



$IM(\Sigma(\Omega(6,3,2)))= N\setminus\{2\}$

Figure 12.

Theorem 3. 4. If n is odd integer greater than 3, then the integer magic spectrum $IM(\Sigma(\Omega(n,n-2,s)))= \cup\, p_i^{ai} N$ where $s = p_1^{ai} p_2^{a2} \ldots p_m^{am}$

Example 9. We show the above result by two examples.

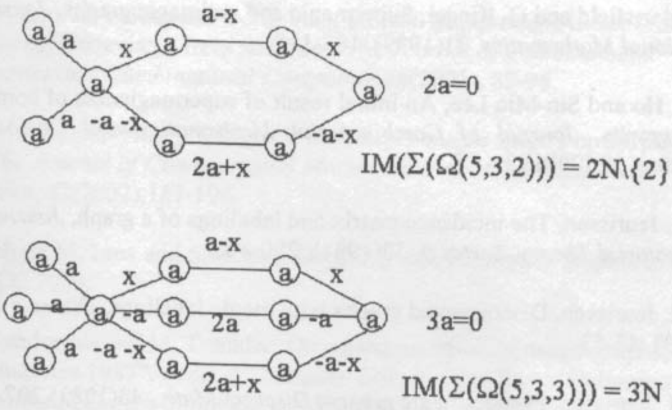


Figure 13.

Theorem 3. 5. If n is even integer greater than 3, then the integer magic spectrum $IM(\Sigma(\Omega(n,n-2,s))) = (\cup p_i^{ai} N)\backslash \{2\}$ where $2s-2 = p_1^{ai} p_2^{a2} \ldots p_m^{am}$

Example 10. The following two examples illustrates the above result.

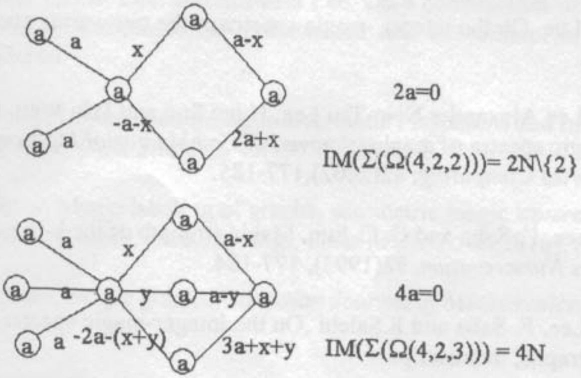


Figure 14.

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