

MAGIC STRENGTH OF THE KTH POWER OF PATHS

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Abstract

A graph is called magic if there exists an assignment $L:E \rightarrow \{1,2,\dots\}$ such that the sum of the labels of all the edges incident to each vertex is the same. We denote the set of all the magic assignments of G by $M(G)$. The strength of a labeling L , denoted by $s(L)$, is defined to be the maximum of $\{L(e):e \in E(G)\}$. The magic strength of the graph is defined to be the minimum of $\{s(L):L \in M(G)\}$.

The k th power of the path P_n , denoted by P_n^k , is the graph resulting from joining every pair of vertices of P_n whose distance from each other is k . The magic strength of P_n^k for various integers of k and n , where $2 \leq k < n$, is presented. In particular, the magic strength of P_n^k is 0 when k and n are both odd.

keywords: magic graph, magic strength, path, regularisable graph, 1-factor of graph, (1,2)-factor of graph.

1. Introduction

Let G be a graph having p vertices and q edges, i.e., if $G=(V,E)$, then $|V(G)|=p$ and $|E(G)|=q$. A labeling, $L:E \rightarrow \{1,2,\dots\}$, is called magic with index w if the sum of the labels of all the edges incident to each vertex is w . Jeurissen [4] called a magic labeling pseudo magic if the labels are pairwise distinct. The concept of magic graph was introduced first by Sedlacek [9, 10], where the label values are real numbers. Stewart [11] considered magic labels L such that $L(e)$ is a set of arithmetic progressions. Doob [2, 3] considered the label values in an abelian group or a ring. Later Lee, Sun, and Kong [7] introduced the concept of magic strength and they defined the strength of a set of magic assignments L , denoted by $s(L)$, to be the maximum of $L(e)$, where $e \in E(G)$. The magic strength of a graph G , denoted by $m(G)$, was defined to be the minimum of $s(L)$ for $L \in M(G)$ [7]. If a graph is not magic, then $m(G)$ is defined to be 0.

Berge [1] called a graph regularisable if a regular multigraph can be obtained from G by adding edges parallel to the edges of G . In fact, a graph G is magic if and only if it is regularisable. Berge showed that the necessary and sufficient condition for a connected non-

bipartite graph G to be regularisable is that $|N(S)| > |S|$ for every nonempty independent set S , where $N(S)$ is the set of neighbors of x in S . Jaeger and Payan [4] characterized regularisable graphs without $K_{1,3}$ as an induced subgraph. In this paper, we consider magic graphs which contain $K_{1,3}$ as an induced subgraph.

For a graph G , a subgraph H is called a 1-factor of G if $V(H)=V(G)$ and $E(H)$ consists of disjoint edges. A cycle of a graph G is a subgraph of G which is connected and regular with degree two. A subgraph H is called a $(1,2)$ -factor of G if $V(H)=V(G)$ and $E(H)$ consists of disjoint edges and cycles. Kocay, Lee, and Wolk [6] showed that a graph G is magic if and only if each edge is contained in some 1-factors of G or $(1,2)$ -factors of G .

Let G^k denote the k th power of graph G for an integer $k > 1$. It is a graph constructed by $V(G^k)=V(G)$ and $E(G^k)$ consisting of all the edges of G and new edges obtained by joining every pair of vertices of G having distance k from each other. The square of a graph which is the second power of G was introduced by Ross and Harary [8]. Let P_n denote the path of order n which is a graph of n ordered vertices and $n-1$ edges, v_i, v_j for $i=2,3,\dots,n$. The P_n^k is the k th power of P_n . A special case of P_n^k is P_n^{n-1} , for $n > 2$, which is a cycle of n vertices.

In this paper, the magic strengths of P_n^k , for $1 < k < n$, are investigated. For the sake of convenience, the graph of P_n^k is drawn in the following manner. (see Figure 1 for $k=4$ and $n=8$) The edges of the path P_n are called the middle edges, the edges v_i, v_j , where $j=i+k$ for $i=1,3,5,\dots$, are called the top edges, and the edges of v_i, v_j , where $j=i+k$ for $i=2,4,6,\dots$, are called the bottom edges. In Section 2, all the cases of $m(P_n^k)=0$ are presented. The magic strengths of P_n^k , for $2 \leq k < 6$, are discussed in Section 3. Finally, a conjecture is proposed in Section 4.

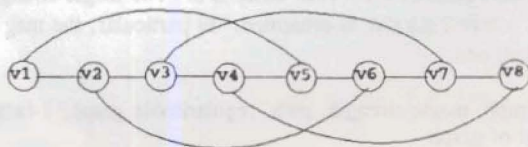


Figure 1 The graph of P_8^4

2. The k th power of paths which are not magic

There are only two groups of power of paths that are not magic, and they are described by the following two theorems.

Theorem 1. $m(P_n^k)=0$ if n and k are both odd and $2 < k < n$.

Proof. We assume, to the contrary, P_n^k is a magic graph with a positive integer index w which is the sum of labels incident to each vertex. The vertices of P_n^k are partitioned into two sets: $I=\{v_1, v_3, \dots, v_n\}$ and $I^*=\{v_2, v_4, \dots, v_{n-1}\}$. (see Figure 2 for $k=3$ and $n=7$) The edges of P_n^k are decomposed into mutually exclusive and exhaustive stars having center at the vertices from either

set I or set I^* . The total labels of edges of P_n^k is $w(n+1)/2$ if set I is selected and is $w(n-1)/2$ if set I^* is selected. Therefore,

$$w(n+1)/2 = w(n-1)/2$$

This shows that $w=0$, which contradicts the assumption.

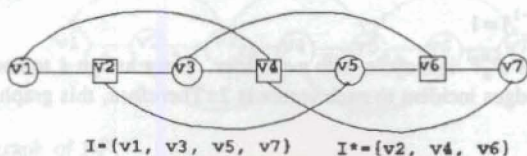


Figure 2 The graph of P_7^3

Theorem 2. $m(P_n^k)=0$ if k and n are both even and $k \geq n/2$.

Proof. We assume, to the contrary, that P_n^k is a magic graph with index w where k and n are even and $k \geq n/2$. The vertices of P_n^k are partitioned into two sets: $I = \{v_1, v_3, \dots, v_{n/2-1}, v_{n/2+2}, v_{n/2+4}, \dots, v_n\}$ and $I^* = \{v_2, v_4, \dots, v_{n/2}, v_{n/2+1}, v_{n/2+3}, \dots, v_{n-1}\}$, each has $n/2$ vertices. (see Figure 3 for $n=8$ and $k=4$). Let d which is a positive integer, represent the label of the edge $v_{n/2}v_{n/2+1}$. Because $k \geq n/2$, there doesn't exist any top edge or bottom edge of $v_i v_j$ for $1 \leq i, j < n/2$ or $n/2 < i, j \leq n$. If the set I is considered, then the edges of P_n^k are decomposed into mutually exclusive stars having center at the vertices from set I and a single edge $v_{n/2}v_{n/2+1}$, and the total labels of edges is $w \cdot n/2 + d$. If the set I^* is considered, then the edges are decomposed into mutually exclusive stars having center at the vertices from set I^* except that the edge $v_{n/2}v_{n/2+1}$ is included twice, and the total labels of edges is $w \cdot n/2 - d$. Therefore,

$$w \frac{n}{2} + d = w \frac{n}{2} - d$$

This shows that $d=0$, which contradicts the assumption.

This theorem can be proved by the following alternative way. Let k and n be even positive

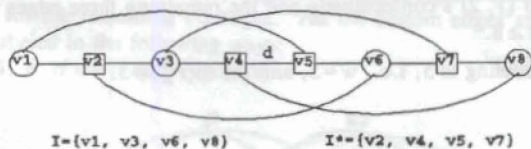


Figure 3 The graph of P_8^4

integers. If $k=n/2$, the edge $v_k v_{k+1}$ is not in a 1-factor or (1,2) factor of P_n^k . If $k > n/2$, the edge $v_2 v_3$ is not in a 1-factor or (1,2)-factor of P_n^k . According to the theorem [6], mentioned in Section 1, the P_n^k for k and n even and $k \geq n/2$ is not magic.

3. The k th power of paths which are magic

Except for the two groups mentioned in the Section 2, we believe that the rest of the power of paths are magic. Some of the P_n^k that are magic, for $2 \leq k \leq 6$, are discussed and the corresponding magic strengths are given in this section.

Theorem 3. $m(P_n^{n-1})=1$.

Proof. The graph of P_n^{n-1} is a cycle with n vertices. If we assign 1 to each edge then the sum of labels of all the edges incident to each vertex is 2. Therefore, this graph is magic with magic

Theorem 4. $m(P_n^k)=2$ if k is odd and $n=2k$.

Proof. Let $k=2t+1$. The top edges from left to right are labeled by 2, $t-1$ 1's and 2 for the last one. All the bottom edges (only on edge for $n=6$) are labeled by 1. The middle edges from left to right are labeled by t (2, 1)'s consecutively, followed by 1, and then t (1, 2)'s consecutively. The index is 4 and the magic strength is 2. (see Figure 4 for $n=10$ and $k=5$)

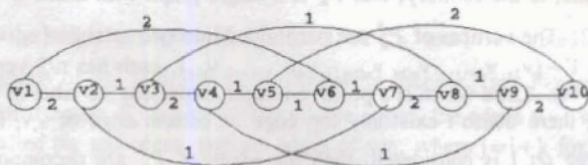


Figure 4 The graph of P_{10}^5

Theorem 5. (1) $m(P_n^2)=3$ if n is even and $n \geq 6$.

(2) $m(P_n^2)=4$ if n is odd and $n \geq 5$.

Proof. (1) For $n \geq 6$ and even, we let $n=2t$ for $t \geq 3$. The top edges from left to right are labeled by 2 and $t-2$ 1's consecutively and the bottom edges from left to right by $t-2$ 1's consecutively and 2 for the last. The middle edges from left to right are labeled in the following ways:

- 3, 1, 1, 1, 3 if $n=6$ (see Figure 5)
- 3, 1, and $t-3$ (1, 2)'s consecutively and the remaining three edges by 1, 1, 3 for $n \geq 8$.

The index of this labeling is 5, i.e., $w=5$, and the $m(P_n^2)=3$.

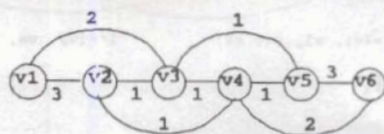


Figure 5 The graph of P_6^2

(2) For $n \geq 5$ and odd, we let $n = 2t + 1$ for $n \geq 2$. The top edges from left to right are labeled by 2, $t-2$ 1's consecutively, and 2 for the last. All the bottom edges are labeled by 1 and the middle edges from left to right are labeled by 4, 1, and $2t-4$ 2's and the remaining two by 1, 4. The index of this labeling is 6 and the magic strength is 4. (see Figure 6 for $n=7$)

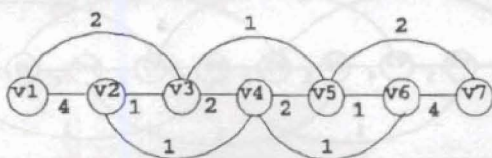


Figure 6 The graph of P_7^2

Theorem 6. $m(P_n^3) = 3$ if n is even and $n \geq 8$.

Proof. All the top edges are labeled by 3 and all the bottom edges are labeled by 1. The middle edges from left to right are labeled by 3, 2 for the first two edges, 2, 3 for the last two edges, and 1 for the remaining $t-5$ edges. The index is 6 and the magic strength is 3. (see Figure 7 for $n=10$)

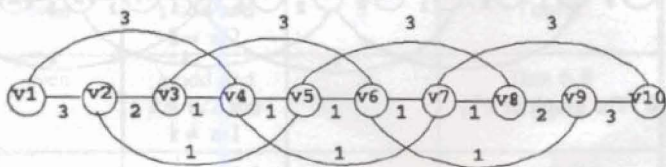


Figure 7 The graph of P_{10}^3

Theorem 7. (1) $m(P_n^4) = 4$ if n is odd and $n \geq 7$.

(2) $m(P_n^4) = 3$ if n is even and $n \geq 10$.

Proof. (1) Let $n = 2t + 1$ for $t \geq 3$. The top edges from left to right are labeled by 2 for the first and the last and 1 for the remaining $t-3$ edges. All the bottom edges are labeled by 1. The middle edges are labeled in the following ways:

1). 4, 1, 3, 3, 1, 4 if $n=7$. (see Figure 8)

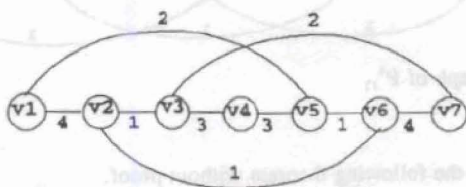


Figure 8 The graph of P_7^4

2). 4, 1, 4, 1, for the first four edges, 1, 4, 1, 4 for the last four edges, and 2 for all the remaining $n-9$ edges. (see Figure 9 for $n=11$)
 The index for this labeling is 6 and the magic strength is 4.

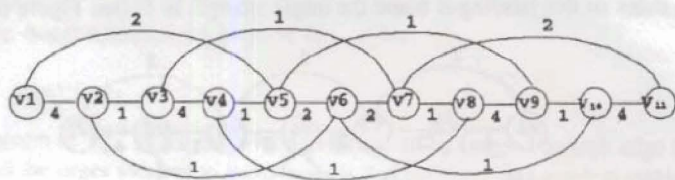


Figure 9 The graph of P_{11}^4

(2) Let $n=2t$ for $t \geq 5$. The top edges from left to right are labeled by 2 for the first and 1 for the rest. The bottom edges from left to right are labeled by 1 for the first $t-3$ edges and 2 for the last. The middle edges from left to right are labeled by 3, 1, 3, 1 consecutively, then $t-5$ (1, 2)'s, and 1, 1, 3, 1, 3 consecutively. (see Figure 10 for $n=12$) The index is 5 and the magic strength is 3.

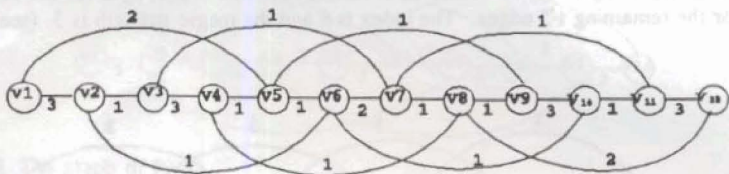


Figure 10 The graph of P_{12}^4

Theorem 8. $m(P_n^5) = 3$ if n is even and $n \geq 8$.

Proof. All the top edges are labeled by 2 and all the bottom edges are labeled by 1. The middle edges from left to right are labeled by 3, 1, 2, 2 consecutively, followed by $n-9$ 1's, and then 2, 2, 1, 3 consecutively. The index is 5 and the magic strength is 3. (see Figure 11 for $n=12$)

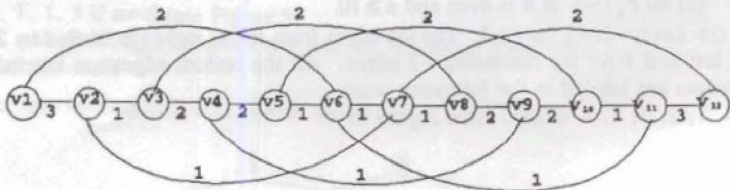


Figure 11 The graph of P_{12}^5

Finally, we state the following theorem without proof.

Theorem 9. (1) $m(P_n^6) = 3$ if n even and $n \geq 14$.

(2) $m(P_n^6) = 4$ if n odd and $n \geq 9$.

4. Summary

The magic strengths of P_n^k for $2 \leq k \leq 6$ were discussed in sections 2 and 3. From the tendency, we can conjecture the magic strengths of P_n^k for $k > 6$ and generalize the results as shown in Table I.

n	k	$m(P_n^k)$	Theorems
Odd	Odd	0	Thm 1
Odd	$K=n-1$	1	Thm 3
Odd	even and $k \neq n-1$	4	Thm 5,7,9
even	$k=n-1$	1	Thm 3
even	k even and $k \geq n/2$	0	Thm 2
even	k odd and $k = n/2$	2	Thm 4
even	k odd and $k \neq n/2$ and $k \neq n-1$	3	Thm 6,8
even	k even and $k < n/2$	3	Thm 5,7,9

TABLE I. Conjecture of Magic Strength of P_n^k

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