On the Integer-Magic Spectra of Honeycomb Graphs

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Abstract

For a positive integer k, a graph $G = (V, E)$ is $\mathbb{Z}_k$-magic if there exists a function, namely, a labeling, $l : E(G) \to \mathbb{Z}_k^*$ such that the induced vertex set labeling $l^+ : V(G) \to \mathbb{Z}_k$, where $l^+(v)$ is the sum of the labels of the edges incident with a vertex v is a constant map. The set of all positive integer k such that G is k-magic is denoted by $\mathrm{IM}(G)$. We call this set the ***integer-magic spectrum*** of G. In this paper, we investigate the integer-magic spectra of the honeycomb graphs. We show that besides $\mathbb{N}$, there are only two types of integer-magic spectra of honeycomb graphs.

1 Introduction

Let A be an abelian group written additively and $A^* = A - \{0\}$. A ***labeling*** is a map l from $E(G)$ to $\mathbb{Z}_k^*$. Given a labeling on the edge set of G, we can induce a vertex set labeling l^+ by adding all the labels of the edges incident with a vertex v, that is,

$$l^+(v) = \sum\{l(u, v) : (u, v) \in E(G)\}.$$

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A graph G is known as ***A*-magic** if there is a labeling $l : E(G) \to A^*$ such that for each vertex v, the sum of the labels of the edges incident with v are all equal to the same constant; i.e., $l^+(v) = c$ for some fixed c in A. We call $\langle G, l \rangle$ an ***A*-magic graph**. In general, a graph G may admit more than one labeling to become an A-magic graph.

We denote the class of all graphs (either simple or multiple graphs) by *Gph* and the class of all abelian groups by *Ab*. For an abelian group A in *Ab*, we also denote the class of all A-magic graphs by${}_AMGp$.

The $\mathbb{Z}$-magic graphs are considered in Stanley [26, 27]. He pointed out in [26] that the theory of magic labelings can be put into the more general context of linear homogeneous diophantine equations.

When the group is $\mathbb{Z}_k$, we shall refer the $\mathbb{Z}_k$ magic graph as ***k*-magic**. The k-magic graphs are studied by [14], [16] and [17].

A-magic graphs are also considered by Doob in [2],[3] and [4] where A is an abelian group. Given a graph G, the problem of deciding whether G admits a magic labeling is equivalent to the problem of deciding whether a set of linear homogeneous Diophantine equation has a solution. (See [23].) At present, no general efficient algorithm is known for finding magic labelings of a graph for a given abelian group.

The original concept of A-magic graph is due to J. Sedlacek [24] and [25], who defines it to be a graph with real-valued edge labeling such that

(i) distinct edges have distinct nonnegative labels, and

(ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

In this paper we use $\mathbb{N}$ to denote the set of natural numbers, $\{1, 2, 3, \ldots\}$. A graph G equipped with a magic labeling $l : E(G) \to \mathbb{N}$ is called $\mathbb{N}$-magic. It is well-known that a graph G is N-magic if and only if each edge of G is contained in a 1-factor (a perfect matching) or a $\{1, 2\}$-factor. (See [9].) Readers shall refer to [5],[6],[7],[8],[12],[14],[24],[25] for $\mathbb{N}$-magic graphs. Note that the $\mathbb{Z}$-magic condition is weaker than $\mathbb{N}$-magic condition. Figure 1 shows a graph which is $\mathbb{Z}$-magic but not $\mathbb{N}$-magic.

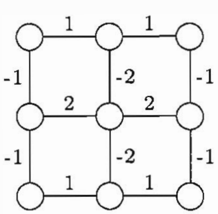

Figure 1: $\mathbb{Z}$-magic but not $\mathbb{N}$-magic.

For convenience, we name $\mathbb{Z}$-magic as 1-magic. For a graph G, the set of all positive integer k such that G is k-magic is denoted by IM(G). We call IM(G) the ***integer-magic spectrum*** of G. In [16], Lee, Sun, Wen and the first author investigate these sets for general graphs . It is shown in [17] that all the grid graph $P_m \times P_n$ has IM$(P_m \times P_n) = \mathbb{N} - \{2\}$, except $P_2 \times P_2$ which is IM$(P_2 \times P_2) = \mathbb{N}$. In fact, a more general result is obtained for polyominoes from square lattices in [22].

A polyomino is a finite connected union of cells on a lattice. Polyominoes are sprung from recreative mathematics domains, from physics such as Ising models, and from polymer of chemistry. In this paper, we consider polyominoes with hexagonal cells, (See Figure 2,) namely, a ***honeycomb graph***. They are called polyhexes in Harary and Read in [5]. These graphs have been studied by chemists as models of the molecular structure of organic compounds build up entirely from benzene rings. We will investigate the integer-magic spectra of honeycomb graphs.

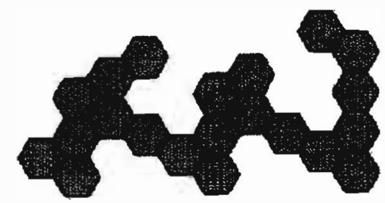

Figure 2: Polyominoes with hexagonal cells

The Honeycomb graphs have been studied from various directions. We put some papers in the reference, even they do not relate to the magic spectrum we studied here. For example, the magic valuation considered in [4] and [13] does not relate to our concept. Papers [15],[16],[18] and[21] deal with more general concept of k-magic graphs.

2 Honeycomb graphs whose dual graphs are trees.

Let H be a honeycomb graph with only one cell. Since H is eulerian with even number of edges, by a result in [21], its integer-magic spectrum is $\mathbb{N}$.

Given a honeycomb graph G, we can associate it with a graph Dual(G) where the vertices of Dual(G) is the set of all hexagonal components and two components A and B are adjacent if they share a common edge.

The graph Dual(G) is called the ***interior dual graph*** of the honeycomb graph G. If we replace the cell of the polyomino by a vertex at the center and join the vertex to its neighbor vertex, then we obtain a dual graph of the polyomino. For the graph H, we can see that Dual(H) is a one vertex graph. Figure 3 illustrates a honeycomb graph and its dual.

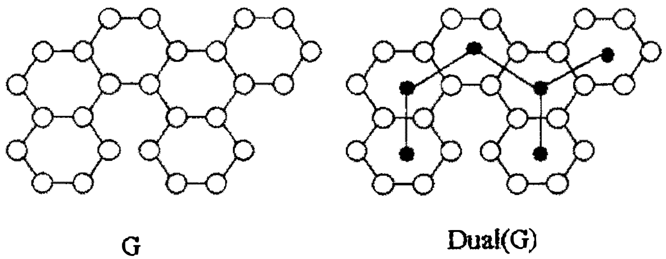

Figure 3: A honeycomb graph and its dual

If a honeycomb graph G has more than one cell, then by definition, its dual graph Dual(H) has more than one vertex.

In [17], it is shown that a graph G is 2-magic if and only if its degree set has the same parity.

Given two honeycomb graphs G_1, G_2 and two specify boundary edges in the same direction, we can form a new honey comb graph by disjoint union them and identify the two boundary edges. (See Figure 4.) We say the new graph is a ***concatenation*** of G_1 and G_2.

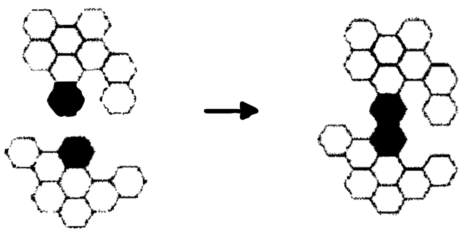

Figure 4: A concatenation of two honeycomb graphs

It is convenient to treat a honeycomb graph as hexagons which have grown, by concatenation, from a single hexagonal cell H. We shall allow an n-cells honeycomb graph to grow into an $(n+1)$-cells honeycomb graph by concatenating an H to an existing boundary edge, subject to the proviso that this does not create a vertex which is common to three hexagons.

Theorem 2.1 *If the interior dual graph of a honeycomb graph G is a non-trivial tree then the integer magic spectrum for G is* $\mathbb{N} - \{2\}$.

Proof. Since G is not regular, its degree sequence consists of 2 and 3.

Therefore, G is not 2-magic. We shall prove that it is k-magic for all k except 2 by induction on the order of the vertices of dual tree of G.

Base Case: If $|V(\mathrm{Dual}(G))| = 2$ then it is of the form of concatenation of two Hs. We can assign a $\mathbb{Z}$-magic labeling on G shown by Figure 5. For all integer $k > 3$, if we replace the -1 and -2 in Figure 5 by $k-1$ and $k-2$, respectively, then the label of each vertex is k. Thus, it becomes a k-magic labeling. Therefore, IM(G) $= \mathbb{N} - \{2\}$.

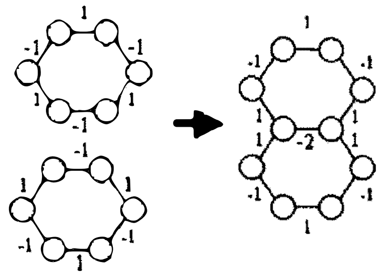

Figure 5:

Induction hypothsis: Assume that all honeycomb graphs G with n vertices in its dual graph have the integer-magic spectrum IM(G) $= \mathbb{N} - \{2\}$. Suppose that G^* is a honeycomb graph with $n+1$ vertices in its dual graph. Since this dual graph is a tree, it is clear that if we delete a leaf of Dual(G^*) then we have a tree of order n. Let G be the honeycomb graph with a associated dual tree. By induction hypothesis it is k-magic for $k \neq 2$.

Since G^* is a concatenation of G and H, we can extended a k-magic labeling of G to G^* as follows: If the identify edge in G has label a then we label the edge on H by label the edge begin on the specify edge of H with a and then $k-a, a, k-a, \ldots$ consecutively along the cycles of H clockwise. (In case $k = 1$, we label $a, -a, a, -a, \ldots$ consecutively).

It is obvious in G^* the identify edges has label $2 \times a$ and all the other edge labels of the remaining edges are same as the origin labeling. We see that each vertex has sum 0. Thus G^* is k-magic. Hence by induction the theorem is true for all honeycomb graphs with dual graph are trees. □

A honeycomb graph is called a ***snake*** if its interior dual graph is a path. Some examples of snake honeycomb graphs are shown in Figure 6.

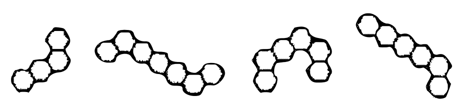

Figure 6: Snake honeycomb graphs

By theorem 2.1, we have

Corollary 2.2 *All snake honeycomb graphs except H have $\mathbb{N}-\{2\}$ as their integer-magic spectra.*

3 A general theorem of honeycomb graphs

For the honeycomb graph H, we adopt the following notation. First, we place H to have two horizontal edges. Then, we label its edges by the sequence $\{a, -a, a, -a, \ldots\}$ in the counterclockwise direction starting from the top horizontal edge. Finally, we put a number a in the center of the hexagonal to present the $\mathbb{Z}$-magic labeling of H. Figure 7 shows some examples. We name the top horizontal edge by e_1 and the rest edges by $e_2, e_3, \ldots, e_6$, respectively, in the counterclockwise direction.

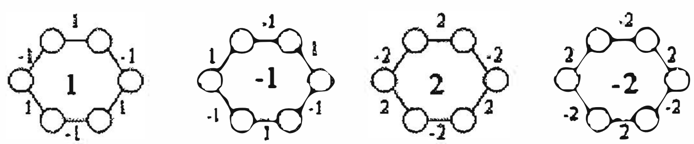

Figure 7: Honeycomb graph notations

This notation allows us to describe the integer-magic labelings of the honeycomb graphs easily. Let a honeycomb graph has two adjacent cells H_a and H_b with labels a and b, respectively, in their centers. If the common edge is one of the e_1, e_3, or e_5 edges of H_a, then the common edge is labeled by $a-b$. Otherwise, the common edge is labeled by $b-a$. The figure 8 illustrates how this notation works for concatenating several hexagons and what the $\mathbb{Z}$-magic labeling is.

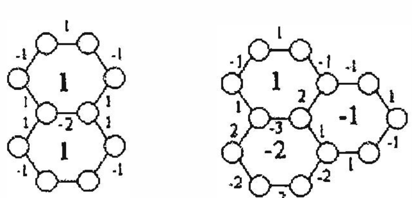

Figure 8: concatenates several hexagons and their $\mathbb{Z}$-magic labelings

When we concatenate two hexagons and label by the above notations, the two edges to concatenate become a common edge. To keep the $\mathbb{Z}$-magic

labeling with sum $s = 0$, we add the values of these two edges and label it to the common edge. The common edge of two cells can only happen between one odd edge and one even edges because of the way we place a honeycomb graph. Therefore, the value of the common edges must be either $a-b$ or $b-a$. Since we simply add the values of two edges together and the sign of the edge is determined by its parity, we know that the labeling is determined by the rule described above.

Now we want to consider honeycomb graphs which are called hexagonal clusters. The following figure 9 demonstrates hexagonal clusters HC(2), HC(3), HC(4), and HC(5).

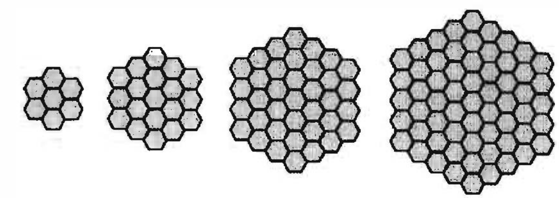

Figure 9: hexagonal clusters HC(2), HC(3), HC(4), HC(5)

Figure 10 provides a $\mathbb{Z}$-magic labeling of a hexagonal cluster HC(2).

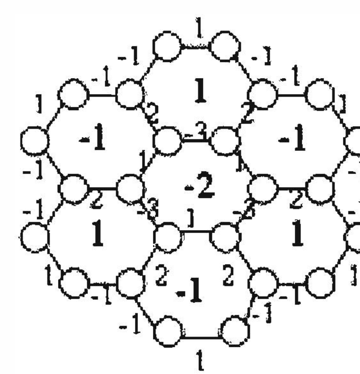

Figure 10: A $\mathbb{Z}$-magic labeling of a hexagonal cluster HC(2)

We use this honeycomb graph to tile the plane which is shown in Figure 11.

We call this tiling an ***infinite universal honeycomb graph*** UH. Since, during tiling, there is no new vertex or edge added, UH is Z-magic. It is clear that any honeycomb graph G is a subgraph of UH. Thus, the

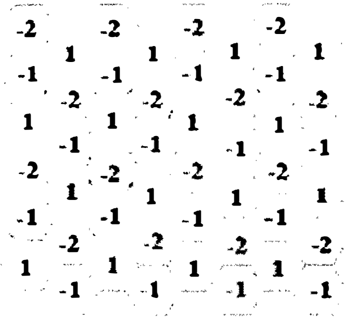

Figure 11: an infinite universal honeycomb graph UH

labeling coming from UH is a $\mathbb{Z}$-magic labeling of G. Figure 12 gives us an example.

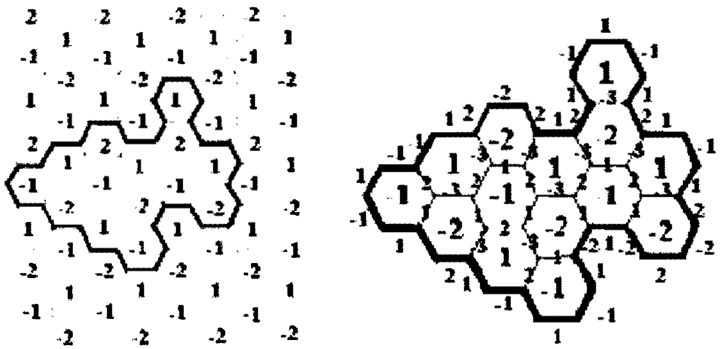

Figure 12: A $\mathbb{Z}$-magic labeling of a honeycomb graph G

To see that the graph is k-magic for all $k > 4$, we replace all the edges with a negative label $-a$ in our base hexagonal cluster HC(2) in figure 10 by $k - a$. Then, the label in a vertex become either k or $2k$ depending on the number of edges which are adjacent to this vertex. Thus, in $\mathbb{Z}_n$, the labels of the vertices all become 0. Therefore, it is a k-magic labeling.

We now summarize the about results and conclude that

Theorem 3.1 *The integer-magic spectrum for any non-trivial honeycomb graph contains* $\mathbb{N} - \{2, 3\}$.

4 Honeycomb graphs which are $\mathbb{Z}_3$-magic

In this section, we determine whether a honeycomb graph is $\mathbb{Z}_3$-magic or not. By definition, there are two types of a $\mathbb{Z}_3$-magic labeling of a graph G with either 0 or 1 as the sum of each vertex, respectively. We say a graph is $\mathbb{Z}_3$-magic with the sum $s = n$ if there is a labeling of this graph with the sum of the adjacent edges equals to n at every vertex.

Lemma 4.1 *A graph G is Z_3-magic with sum $s = 1$ if and only if it is a $\mathbb{Z}_3$-magic with sum $s = -1$.*

Proof. If G is a $\mathbb{Z}_3$-magic with sum $s = 1$, then we can obtain another labeling by multiply $V1$ and get a $\mathbb{Z}_3$-magic labeling with sum $s = -1$ and vise versa. □

The following figures demonstrate different cases of Z_3-magic graphs.

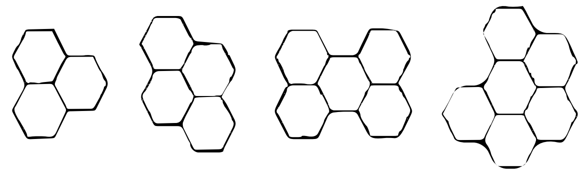

Figure 13: Graphs are not Z_3-magic.

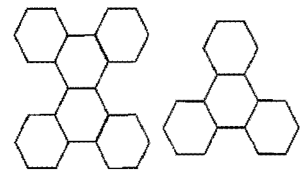

Figure 14: Z_3-magic with $s = 1$ and $s = 0$

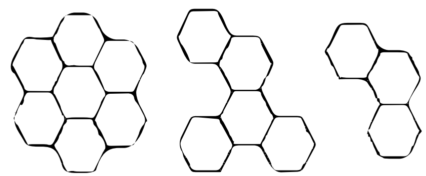

Figure 15: Z_3-magic with $s = 0$ but not $s = 1$.

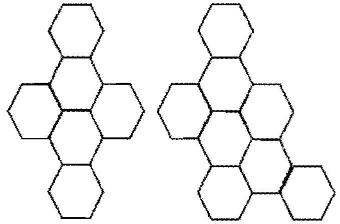

Figure 16: Z_3-magic with $s = 1$ but not $s = 0$

Before the further investigation, we list two rules about the labeling on the incident edges to a given vertex in a $\mathbb{Z}_3$-magic graph. In the other words, for a $\mathbb{Z}_3$-magic graph, the labeling on each edges incident to a vertex must satisfy one of the following rules:

- **Rule 1a** For the vertex with degree 2, the $\mathbb{Z}_3$-labeling with sum $s = 0$ at this vertex should be as in the figure 13 a.
- **Rule 1b**For the vertex with degree 2, the $\mathbb{Z}_3$-labeling with $s = 1$ at this vertex should be as in the figure 13b.
- **Rule 2a** For the vertex with degree 3, the $\mathbb{Z}_3$-labeling with $s = 0$ at the vertex should be as in the figure 14a.
- **Rule 2b** For the vertex with degree 3, the $\mathbb{Z}_3$-labeling with sum $s = 1$ at the vertex should be as in the figure 14b.

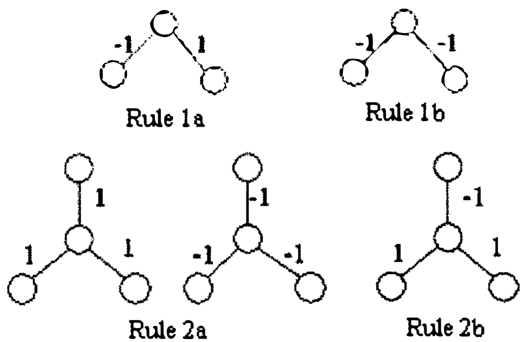

Figure 17: $\mathbb{Z}_3$-labeling Rules

For a honeycomb graph G, we delete all the vertices of degree 3 and their incident edges and denote this subgraph by RD$(G, 2)$. Similarly, we delete

all the vertices of degree 2 and their incident edges and denote this subgraph by RD$(G, 3)$. We call RD$(G, 2)$ and RD$(G, 3)$ the ***reduced graphs*** of a given graph G with all its vertices of degree 2 and 3, respectively. Figure 18 shows an example of RD$(G, 2)$ and RD$(G, 3)$.

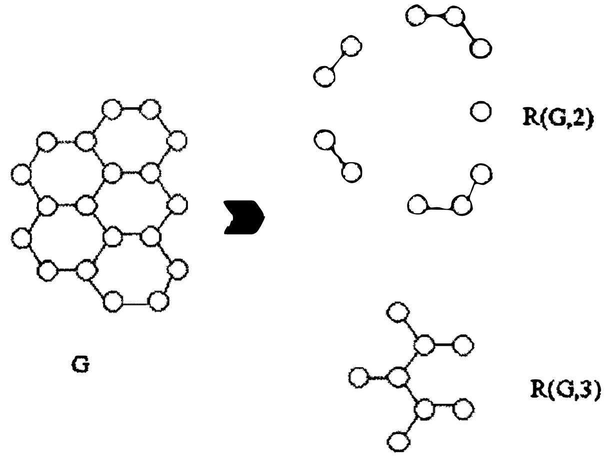

Figure 18: RD$(G, 2)$ and RD$(G, 3)$

Lemma 4.2 *Let G be a honeycomb graph with a Z_3-labeling with sum $s = 0$. All the edges of a connected subgraph of RD$(G, 3)$ must be labeled by the same number.*

Proof. By rule 2a, we can only label the edges of a vertex of order 3 by either all 1 or all -1. There are only vertex of order 3 in the RD$(G, 3)$. In a connected subgraph of RD$(G, 3)$, every vertex shares an edge with at least one other vertex. Thus, all edges in a connected subgraph of RD$(G, 3)$ must be labeled by the same number. □

For a honeycomb graph G, a *bridge cell* is a hexagonal cell of G such that it has only two connected hexagonal cells with two opposite edges.

A bridge cell contains two opposite vertices of order 2 and the two edges of a vertex of order 2 must be labeled differently by the rule 1a. We place two vertices of order 2 in a horizontal line and then we can split the rest of four vertices of order 3 into top and bottom sides. Each side contains two vertices of order 3 which share a common edge. By Lemma 4.2, the edges of two vertices of order 2 which are adjacent to the top side must be labeled by the same number. Similarly, the two edges of two vertices of order 2 which are adjacent to the bottom side must be labeled by the same number which is different than then left side one. Thus, in a Z_3-labeling with sum $s = 0$,

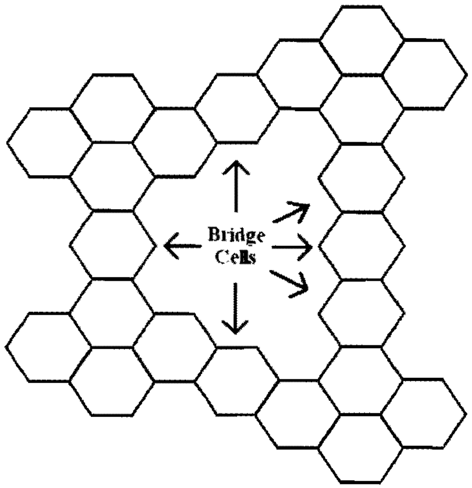

Figure 19: Bridge Cells

a bridge cell must connect two distinct connected subgraphs of RD(G, 3) with different numbers on the edges. (See Figure 20.)

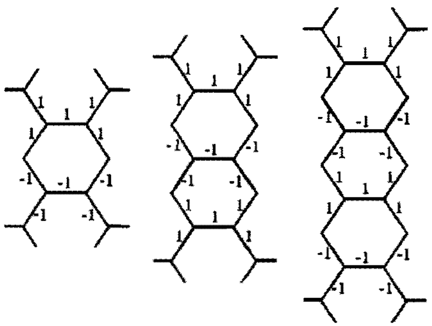

Figure 20: Bridge Cells Labeling

If two distinct connected subgraphs of RD(G, 3) are connected by two distinct chains of bridge cells, we can remove one and these two subgraphs remain connected in G as long as the lengths of these two chains have the same parity. These bridge cells do not affect the possibility of Z_3-labeling with sum $s = 0$. Thus, we can assume that every two distinct connected subgraphs of RD(G, 3) are connected by only one bridge cell.

After removing extra bridge cells, we can treat a connected subgraphs of RD(G, 3) as a vertex and a bridge cell as an edge adjacent two vertices, namely, the *simplified graph* of G. (See Figure 21.) In the simplified graph of G, every edge must connect two vertices which are labeled differently. Thus, every path in the simplified graph of G can be labeled by 1 and -1 alternatively. This means a $\mathbb{Z}_3$-labeling with sum $s = 0$. Therefore, we only need to consider a cycle in the simplified graph of G.

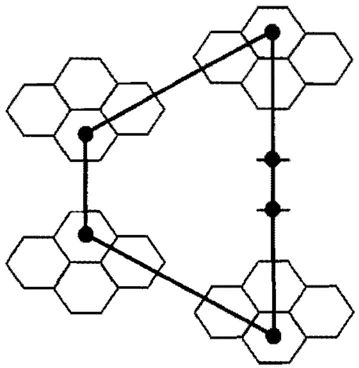

Figure 21: Example of a Simplified Graph

Theorem 4.3 *A honeycomb graph G would not be $\mathbb{Z}_3$-magic with sum $s =$ 0 if and only if there is a cycle C_n with n is odd in the simplified graph of G.*

Proof. A cycle C_n with n is odd cannot be labeled by 1 and -1 alternatively. But, a cycle C_n with n is even can. □

Thus, we only need to focus on a honeycomb graph G with a connected RD(G, 3) subgraph. By Lemma 4.2, without loss of generality, we also assume that all the edges of the vertices of order 3 are labeled by 1.

Theorem 4.4 *A honeycomb graph G with a connected RD(G, 3) would not be $\mathbb{Z}_3$-magic with sum $s = 0$ if and only if there is a P_3 or a vertex of order 0 in the reduced graph RD(G, 2)*

Proof. The both side vertices of P_3 have incident edges which are also incident to a degree 3 vertex. By the rule 2a, it force the two edges of P_3 must be labeled by the same values 1. But it violates the rule 1a. The same argument works for a vertex of order 0.

Conversely, in RD(G, 2), any subgraph which is not a path contains a vertex of order 3. Also, in RD(G, 2), any path of order more than 5 is either coming from two adjacent hexagonal cells or it is contains in a single hexagonal cell H. Since our G is not single hexagonal cell H, there is no path of order more than 5 in $\mathrm{RD}(G,2)$ since any path of order more than 5 must contain a common edge of two adjacent hexagonal cells. So, if there is no P_3 nor a vertex of order 0 in the reduced graph $\mathrm{RD}(G,2)$, then the only two possible subgraphs in $\mathrm{RD}(G,2)$ are P_2 and P_4. For a P_2, we label the edge by -1. For a P_4, we label the there edges by -1, 1 and -1. Then, we have a $\mathbb{Z}_3$-magic labeling with sum $s=0$. □

Theorem 4.5 *A honeycomb graph G would not be $\mathbb{Z}_3$-magic with $s=1$ if and only if there is a vertex with degree 1 in the reduced graph $RD(G,3)$.*

Proof. If there is a vertex of degree 1 in the reduced graph $\mathrm{RD}(G,3)$, then, by rule 2b, we know that all edges incident to a degree-2 vertex must be labeled by -1. The degree-1 vertex in $\mathrm{RD}(G,3)$ has the other two incident edges labeled by -1 in the original graph G. It violates the rules 2b.

Conversely, we assume that there are no vertices of degree 1 in the reduced graph $\mathrm{RD}(G,3)$. Notice that the only possible honeycomb graph whose vertices are all degree 3 is the universal honeycomb graph. Thus, we have some degree 2 vertices in the reduced graph $\mathrm{RD}(G,3)$ and it must be in the boundary of the reduced graph $\mathrm{RD}(G,3)$. We can label the reduced graph $\mathrm{RD}(G,3)$ by

1. 1 if an edge is incident to an order 2 vertex,
2. -1 if an edge is on the top or bottom of a honeycomb cell,
3. 1 if an edge is on the sides of a honeycomb cell.

(See Figure 22.)

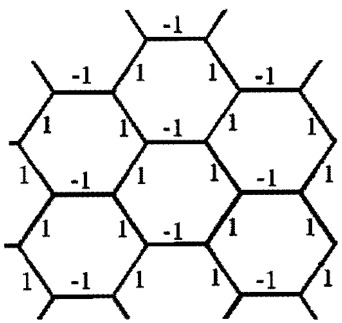

Figure 22: How to label inside degree 3 vertex for $s=1$.

Note here that all the honeycomb cells in a honeycomb graph have two horizontal edges on the top and the bottom and four edges on the side.

Since, in a honeycomb cell, a degree 3 vertex is incident to one horizontal edge and two sided edges. According to our labeling, the sum of a degree 3 vertex is $(-1)+1+1=1$. For a degree 2 vertex in the reduced graph $\mathrm{RD}(G,3)$, its two edges are labeled by 1. So, this vertex, as a degree 3 vertex in G, has the sum 1 since the third edge is labeled by -1 which is incident to a degree 2 vertex in G. Also, by rule 2b, we know that all edges incident to a degree-2 vertex must be labeled by -1. Therefore, we have a Z_3-magic labeling with $s=1$ for the honeycomb graph G.

We complete the proof. □

Corollary 4.6 *The integer magic spectrum for any hexagonal cluster $HC(n)$ is*

$$IM(HC(n)) = \begin{cases} \mathbb{N}-\{2\} & \text{if } n=2, \text{ and} \\ \mathbb{N}-\{2,3\} & \text{if } n>2. \end{cases}$$

The figure 23 shows that HC(2) is $\mathbb{Z}_3$-magic labeling with $s=0$.

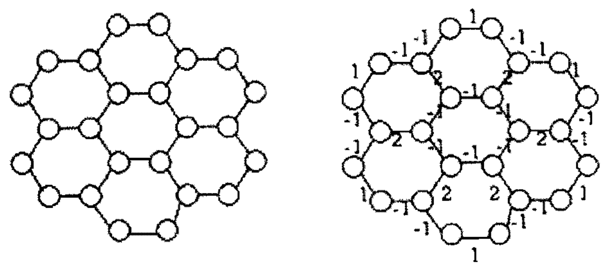

Figure 23:

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