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On friendly index sets of 2-regular graphs

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Abstract

Let G be a graph with vertex set V and edge set E, and let A be an abelian group. A labeling $f : V \to A$ induces an edge labeling $f^* : E \to A$ defined by $f^*(xy) = f(x) + f(y)$. For $i \in A$, let $v_f(i) = \text{card}\{v \in V : f(v) = i\}$ and $e_f(i) = \text{card}\{e \in E : f^*(e) = i\}$. A labeling f is said to be A-friendly if $|v_f(i) - v_f(j)| \le 1$ for all $(i, j) \in A \times A$, and A-cordial if we also have $|e_f(i) - e_f(j)| \le 1$ for all $(i, j) \in A \times A$. When $A = \mathbb{Z}_2$, the friendly index set of the graph G is defined as $\{|e_f(1) - e_f(0)| : \text{the vertex labeling } f \text{ is } \mathbb{Z}_2\text{-friendly}\}$. In this paper we completely determine the friendly index sets of 2-regular graphs. In particular, we show that a 2-regular graph of order n is cordial if and only if $n \not\equiv 2 \pmod 4$.

Keywords: Vertex labeling; Friendly labeling; Cordiality; Friendly index set; Cycle; 2-regular graph

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let A be an abelian group. A labeling $f : V(G) \to A$ induces an edge labeling $f^* : E(G) \to A$ defined by $f^*(xy) = f(x) + f(y)$ for each edge $xy \in E(G)$. For $i \in A$, let $v_f(i) = \text{card}\{v \in V(G) : f(v) = i\}$ and $e_f(i) = \text{card}\{e \in E(G) : f^*(e) = i\}$. A labeling f of a graph G is said to be ***A-friendly*** if $|v_f(i) - v_f(j)| \le 1$ for all $(i, j) \in A \times A$. If, in addition to being A-friendly, we also have $|e_f(i) - e_f(j)| \le 1$ for each $(i, j) \in A \times A$, then f is said to be ***A-cordial***.

The notion of A-cordial labelings was first introduced by Hovey [10], who generalized the concept of cordial graphs of Cahit [2,3]. Cahit considered $A = \mathbb{Z}_2$ and he proved the following: every tree is cordial; K_n is cordial if and only if $n \le 3$; $K_{m,n}$ is cordial for all m and n; the wheel $W_n = K_1 + C_{n-1}$ is cordial if and only if $n \not\equiv 0 \pmod 4$; C_n is cordial if and only if $n \not\equiv 2 \pmod 4$; and an Eulerian graph is not cordial if its size is congruent to 2 (mod 4). Benson and Lee [1] showed a large class of cordial regular windmill graphs which include the friendship graphs as a subclass.

Lee and Liu [15] investigated cordial complete k-partite graphs. Kuo, Chang and Kwong [14] determined all m and n for which mK_n is cordial. Cubic graphs are 3-regular graphs. In 1989, the second author, Ho and Shee [9] completely characterized cordial generalized Petersen graphs. Ho, Lee and Shee [8] investigated the construction of

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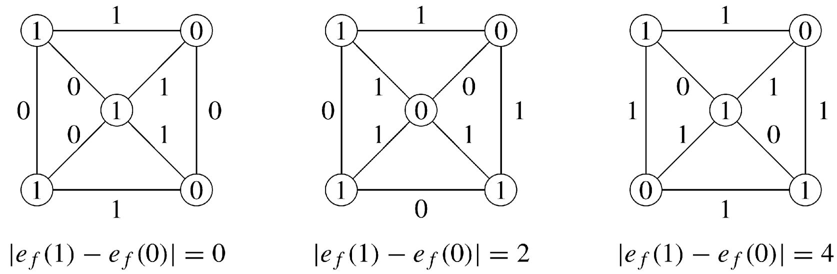

Fig. 1. Friendly labelings of W_5.

cordial graphs by Cartesian product and composition. Seoud and Abdel Maqsoud [19] proved that certain cylinder graphs are cordial. Several constructions of cordial graphs were proposed in [11–13,17–21]. For more details of the known results and open problems on cordial graphs, see [4,7].

In this paper, we will exclusively focus on $A = \mathbb{Z}_2$, and drop the reference to the group. In [6] the following concept was introduced.

Definition 1. The ***friendly index set*** $\mathrm{FI}(G)$ of a graph G is defined as the set $\{|e_f(1) - e_f(0)| : f \text{ is a friendly vertex labeling}\}$.

When the context is clear, we will drop the subscript f. Note that if 0 or 1 is in $\mathrm{FI}(G)$, then G is cordial. Thus the concept of friendly index sets could be viewed as a generalization of cordiality.

Cairnie and Edwards [5] have determined the computational complexity of cordial labeling and $\mathbb{Z}_k$-cordial labeling. They proved this to decide whether a graph that admits a cordial labeling is NP-complete. Even the restricted problem of deciding whether a connected graph of diameter 2 has a cordial labeling is NP-complete. Thus in general it is difficult to determine the friendly index sets of graphs.

In [16] the friendly index sets of a few classes of graphs, in particular, complete bipartite graphs and cycles are determined. The following result was established.

Theorem 1. *For any graph with q edges, the friendly index set $FI(G) \subseteq \{0, 2, 4, \ldots, q\}$ if q is even and $FI(G) \subseteq \{1, 3, \ldots, q\}$ if q is odd.*

Example 1. The graph W_n of order n contains a cycle of order $n - 1$, and for which every graph vertex in the cycle is connected to one other graph vertex. Thus $W_n = K_1 + C_{n-1}$. Fig. 1 illustrates the friendly index set of wheel W_5. □

Example 2. $\mathrm{FI}(K_{3,3}) = \{1, 9\}$ and $\mathrm{FI}(C_3 \times K_2) = \{1, 3, 5\}$. See Fig. 2. □

The second and third authors proposed the following.

Conjecture A. *The numbers in $FI(T)$ for any tree T form an arithmetic progression.*

In [16], it was shown that

Theorem 2. *The friendly index set of a cycle is given as follows:*

$$FI(C_n) = \begin{cases} \{0, 4, 8, \ldots, n\} & \text{if } n \equiv 0 \pmod 4, \\ \{2, 6, 10, \ldots, n\} & \text{if } n \equiv 2 \pmod 4, \\ \{1, 3, 5, \ldots, n-2\} & \text{if } n \text{ is odd}. \end{cases}$$

Thus the numbers in $\mathrm{FI}(G)$ for any cycle G form an arithmetic progression. In this paper we describe the friendly index sets of 2-regular graphs. Denote by $C(n_1, n_2, \ldots, n_k)$ the union of k disjoint cycles of length $n_1, n_2, \ldots, n_k$ respectively. Due to symmetry, we assume $3 \le n_1 \le n_2 \le \cdots \le n_k$. For unions of two cycles, the friendly index sets consist of arithmetic progressions. However, this is not always true when the union contains more than two cycles.

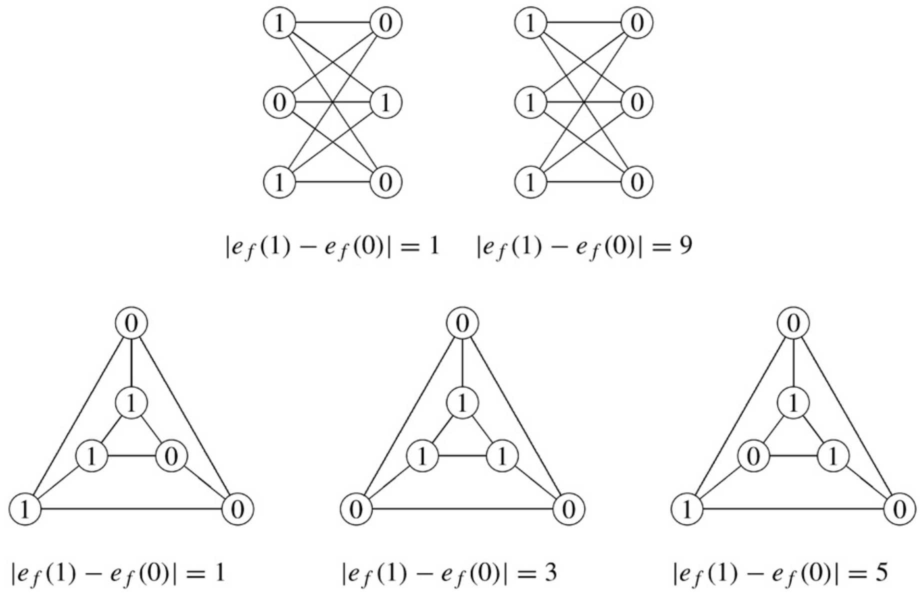

Fig. 2. Friendly labelings of $K_{3,3}$ and $C_3 \times K_2$.

2. Friendly index sets of union of two cycles

We start our investigation by studying the special case of $k = 2$.

Theorem 3. *For any integers n_1 and n_2 satisfying $3 \le n_1 \le n_2$, define*

$$S = \begin{cases} \{0, 4, 8, \ldots, n_1 + n_2\} & \text{if } n_1 + n_2 \equiv 0 \pmod 4, \\ \{2, 6, 10, \ldots, n_1 + n_2\} & \text{if } n_1 + n_2 \equiv 2 \pmod 4, \\ \{1, 3, 5, \ldots, n_1 + n_2\} & \text{if } n_1 + n_2 \equiv 1 \pmod 2. \end{cases}$$

If either $|n_1 - n_2| \le 1$ or both n_1 and n_2 are even, then $FI(C(n_1, n_2)) = S$; otherwise, $FI(C(n_1, n_2)) = S - \{n_1 + n_2\}$.

Proof. For brevity, vertices labeled 0 will be referred to as 0-vertices, and vertices labeled 1 will be called 1-vertices. Likewise, an edge is a 0-edge if its induced edge label is 0, otherwise it is called a 1-edge.

Assume C_{n_1} consists of a block of c_{11} consecutive 0-vertices, followed by a block of d_{11} consecutive 1-vertices, then a block of c_{12} consecutive 0-vertices, then a block of d_{12} consecutive 1-vertices, and so forth; and assume that there are b_1 pairs of such consecutive 0- and 1-blocks in C_{n_1}. If all the vertices are labeled with a constant (either 0 or 1), we assume $b_1 = 0$.

The edges within each block are obviously 0-edges, and 1-edges occur only between two adjacent blocks. Hence $2b_1$ edges of C_{n_1} are 1-edges, and the remaining $n_1 - 2b_1$ edges are 0-edges. If C_{n_2} has b_2 pairs of adjacent 0- and 1-blocks, then the number of 1- and 0-edges in C_{n_2} will be $2b_2$ and $n_2 - 2b_2$ respectively. Therefore $e_f(1) = 2(b_1 + b_2)$ and $e_f(0) = n_1 + n_2 - 2(b_1 + b_2)$; hence $e_f(1) - e_f(0) = 4(b_1 + b_2) - (n_1 + n_2)$. It follows immediately from $e_f(1) - e_f(0) \equiv -(n_1 + n_2) \pmod 4$ that $\mathrm{FI}(C(n_1, n_2)) \subseteq S$.

If $n_1 + n_2 \in \mathrm{FI}(C(n_1, n_2))$, then all the edges in $C(n_1, n_2)$ are either 0-edges or 1-edges. If all the edges are 0-edges, all the vertices within the same cycle must be assigned the same label. In order for $C(n_1, n_2)$ to be friendly, we need $|n_1 - n_2| \le 1$, and label the vertices of one cycle with 0, and the vertices of the other cycle with 1. If all the edges are 1-edges, the vertices in both cycles must be labeled alternately with 0 and 1, and there must be an even number of vertices in both cycles so that no adjacent vertices would be labeled the same, for otherwise a 0-edge would have been formed.

Assume that the sizes of the 0-blocks in C_{n_2} are $c_{21}, c_{22}, \ldots, c_{2b_2}$, and that the 1-blocks of C_{n_2} are of sizes $d_{21}, d_{22}, \ldots, d_{2b_2}$. It remains to show that for any $b_1 + b_2 \ge 1$ within the proper range, there exists a friendly vertex

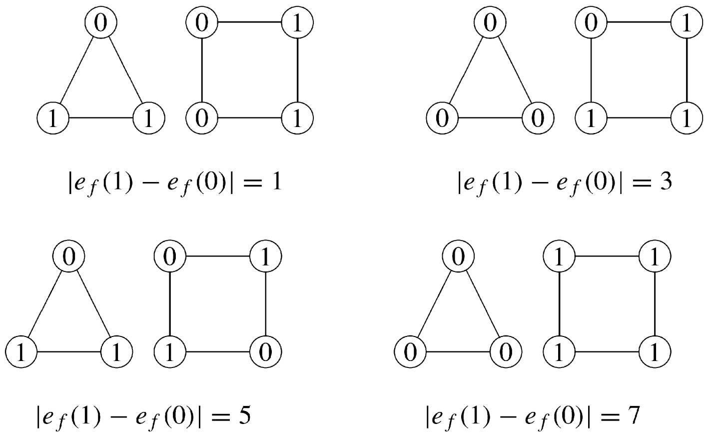

Fig. 3. Friendly labelings of $C(3, 4)$.

labeling that gives $e_f(1) - e_f(0) = 4(b_1 + b_2) - (n_1 + n_2)$. Define

$$c_{1j} = d_{1j} = 1 \quad \text{if } j < b_1,$$
$$c_{2j} = d_{2j} = 1 \quad \text{if } j < b_2,$$
$$c_{1b_1} = \lfloor (n_1 - 2(b_1 - 1))/2 \rfloor, \qquad d_{1b_1} = \lceil (n_1 - 2(b_1 - 1))/2 \rceil,$$
$$c_{2b_2} = \lceil (n_2 - 2(b_2 - 1))/2 \rceil, \qquad d_{2b_2} = \lfloor (n_2 - 2(b_2 - 1))/2 \rfloor.$$

It is obvious that the resulting vertex labeling is friendly if $b_1, b_2 \geq 1$. This covers all the values in S except $|4 - (n_1 + n_2)|$.

If $n_1 < n_2$, we can label all the vertices in C_{n_1} and any $\lceil (n_1 + n_2)/2 \rceil - n_1$ consecutive vertices in C_{n_2} with 0, and the remaining $\lfloor (n_1 + n_2)/2 \rfloor$ vertices in C_{n_2} with 1. This produces a friendly labeling of $C(n_1, n_2)$ with $b_1 = 0$ and $b_2 = 1$, hence $e_f(1) - e_f(0) = 4 - (n_1 + n_2)$. If $n_1 = n_2$, we note that we can choose $b_1 + b_2 \geq 1$ such that $b_1 + b_2 = n_1 - 1 = n_2 - 1$, which leads to $e_f(1) - e_f(0) = (n_1 + n_2) - 4$. The proof is now complete. □

Example 3. Assume the vertices in C_{n_1} and C_{n_2} are $u_1, u_2, \ldots, u_{n_1}$ and $v_1, v_2, \ldots, v_{n_2}$ respectively. The friendly labelings of $C(3, 4)$ displayed below show that $\mathrm{FI}(C(3, 4)) = \{1, 3, 5, 7\}$.

b_1	b_2	u_1	u_2	u_3	v_1	v_2	v_3	v_4	$e(1) - e(0)$
0	0	0	0	0	1	1	1	1	−7
0	1	0	0	0	0	1	1	1	−3
1	1	0	1	1	0	0	1	1	1
1	2	0	1	1	0	1	0	1	5

See Fig. 3. □

Example 4. The tables below depict the friendly labelings of $C(4, 10)$ and $C(5, 10)$.

b_1	b_2	u_1	u_2	u_3	u_4	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	$e(1) - e(0)$
0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	−10
1	1	0	0	1	1	0	0	0	0	0	1	1	1	1	1	−6
1	2	0	0	1	1	0	1	0	0	0	0	1	1	1	1	−2
2	5	0	1	0	1	0	1	0	1	0	1	0	1	0	1	−14

b_1	b_2	u_1	u_2	u_3	u_4	u_5	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	$e(1)-e(0)$
0	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	−11
1	1	0	0	1	1	1	0	0	0	0	0	1	1	1	1	1	−7
1	2	0	0	1	1	1	0	1	0	0	0	0	1	1	1	1	−3
1	3	0	0	1	1	1	0	1	0	1	0	0	0	1	1	1	1
1	4	0	0	1	1	1	0	1	0	1	0	1	0	0	1	1	5
1	5	0	0	1	1	1	0	1	0	1	0	1	0	1	0	1	9
2	5	0	1	0	1	1	0	1	0	1	0	1	0	1	0	1	13

Thus $\mathrm{FI}(C(4, 10)) = \{2, 6, 10, 14\}$ and $\mathrm{FI}(C(5, 10)) = \{1, 3, 5, 7, 9, 11, 13\}$. □

Corollary 4. *For any integers n_1 and n_2 satisfying $3 \le n_1 \le n_2$, the 2-regular graph $C(n_1, n_2)$ is cordial if and only if $n_1 + n_2 \not\equiv 2 \pmod 4$.*

3. Friendly index sets of 2-regular graphs

One may expect Theorem 3 can be naturally extended to unions of more than two cycles. In particular, one may conjecture that $\mathrm{FI}(C(n_1, n_2, \ldots, n_k))$ consists of an arithmetic progression. Unfortunately, it is not always true.

Example 5. The 2-regular graph $C(3, 3, 3, 3)$ contains four 3-cycles. The argument we used in the proof of Theorem 3 shows that $e_f(1) - e_f(0) \equiv -12 \equiv 0 \pmod 4$. Thus $\mathrm{FI}(C(3, 3, 3, 3)) \subseteq \{0, 4, 8, 12\}$. The values 0, 4 and 12 are attainable, as illustrated in the following table.

f	u_1	u_2	u_3	v_1	v_2	v_3	w_1	w_2	w_3	x_1	x_2	x_3	$e(1)-e(0)$
f_1	0	0	0	0	1	1	0	1	1	0	1	1	0
f_2	0	0	1	0	1	1	0	0	1	0	1	1	4
f_3	0	0	0	1	1	1	0	0	0	1	1	1	−12

Note that 8 is missing. In the proof of Theorem 3, we have shown that, in any cycle, $e(1)$ must be even. It follows that $e(1)$ in a union of cycles is also even. Thus the maximum value of $e(1)$ in $C(3, 3, 3, 3)$ is 8. If 8 were in $\mathrm{FI}(C(3, 3, 3, 3))$, then $e(0) = 10$ and $e(1) = 2$, and both 1-edges would be in the same cycle. In this cycle, $|v(1) - v(0)| = 1$. In all the other cycles, $|v(1) - v(0)| = 3$. Such a vertex labeling is not friendly. We conclude that $\mathrm{FI}(C(3, 3, 3, 3)) = \{0, 4, 12\}$. □

Example 6. The 2-regular graph $C(3, 3, 3, 4)$, contains three 3-cycles and one 4-cycle. We find $e_f(1) - e_f(0) \equiv -13 \equiv -1 \pmod 4$. The friendly index set contains 1, 3, 5, 7, 9 and 13; see the following table.

f	u_1	u_2	u_3	v_1	v_2	v_3	w_1	w_2	w_3	x_1	x_2	x_3	x_4	$e(1)-e(0)$
f_1	0	0	1	0	1	1	1	1	1	0	0	0	1	−1
f_2	0	0	1	0	1	1	0	0	1	0	0	1	1	3
f_3	0	0	1	0	1	1	0	0	0	1	1	1	1	−5
f_4	0	0	1	0	1	1	0	0	1	0	1	0	1	7
f_5	0	0	1	1	1	1	1	1	1	0	0	0	0	−9
f_6	0	0	0	1	1	1	0	0	0	1	1	1	1	−13

Note that 11 is missing. Since $e(1)$ is even, we see that the maximum value of $e(1)$ in $C(3, 3, 3, 4)$ is 10. If 11 were in $\mathrm{FI}(C(3, 3, 3, 4))$, then $e(1) = 12$ and $e(0) = 1$, which again contradicts the lemma. Hence $\mathrm{FI}(C(3, 3, 3, 4)) = \{1, 3, 5, 7, 9, 13\}$. □

To find the general solution, we need a careful analysis of the possible values that $e_f(1) - e_f(0)$ could attain. The following notion was introduced in [22].

Definition 2. Let f be a friendly vertex label of a graph, its ***friendly index*** is defined as $i_f = e_f(1) - e_f(0)$. The ***full friendly index set*** FFI(G) of a graph G is the set $\{e_f(1) - e_f(0) : f \text{ is a friendly vertex labeling}\}$.

Adopting the same notations we used in the last section, we group the vertices in each cycle C_{n_i} into $2b_i$ blocks of consecutive 0- and 1-vertices of size $c_{i1}, d_{i1}, c_{i2}, d_{i2}, \ldots, c_{ib_i}, d_{ib_i}$ respectively. If all the vertices in C_{n_i} are labeled the same, define $b_i = 0$. It is clear that $0 \le b_i \le \lfloor n_i/2 \rfloor$. Restricting to C_{n_i}, we find $e_f(1) - e_f(0) = 4b_i - n_i$. Therefore, over $C(n_1, n_2, \ldots, n_k)$, we have $i_f = 4\sum_{i=1}^k b_i - \sum_{i=1}^k n_i$.

Our problem can now be restated as follows. Let $n = \sum_{i=1}^k n_i$, and assume there are ℓ odd numbers among $n_1, n_2, \ldots, n_k$. We want to determine which friendly indices

$$i_f = 4b - n, \qquad 0 \le b \le \sum_{i=1}^k \lfloor n_i/2 \rfloor = \lfloor n/2 \rfloor - \lfloor \ell/2 \rfloor,$$

are attainable by finding an ordered k-tuple $(b_1, b_2, \ldots, b_k)$, where $0 \le b_i \le \lfloor n_i/2 \rfloor$ for each i, that gives $b = \sum_{i=1}^k b_i$ for any specific b within the range.

Before we examine which values of b are attainable, we note that $i_f = 4b - n$ covers the same friendly indices in C_n. Hence FFI($C(n_1, n_2, \ldots, n_k)$) $\subseteq$ FFI(C_n). However, $4(\lfloor n/2 \rfloor - \lfloor \ell/2 \rfloor) - n = n$ if and only if $\ell = 0$. In fact,

$$4\lfloor n_i/2 \rfloor - n_i = \begin{cases} n_i & \text{if } n_i \text{ is even,} \\ n_i - 2 & \text{if } n_i \text{ is odd.} \end{cases}$$

This immediately shows that $n \in$ FFI($C(n_1, n_2, \ldots, n_k)$) if and only if all n_i's are even. More importantly, since $b \le \lfloor n/2 \rfloor - \lfloor \ell/2 \rfloor$, we find

$$i_f \le \begin{cases} n - 4\lfloor \ell/2 \rfloor & \text{if } n \text{ is even,} \\ n - 4\lfloor \ell/2 \rfloor - 2 & \text{if } n \text{ is odd.} \end{cases}$$

Comparing this to

$$\text{FFI}(C_n) = \begin{cases} \{\ldots, n-8, n-4, n\} & \text{if } n \text{ is even,} \\ \{\ldots, n-10, n-6, n-2\} & \text{if } n \text{ is odd,} \end{cases}$$

we observe that FFI($C(n_1, n_2, \ldots, n_k)$) does not contain the last $\lfloor \ell/2 \rfloor$ values in FFI(C_n).

For $b \ge k$, we can easily pick $b_i \ge 1$ for each i such that $b = \sum_{i=1}^k b_i$, and label the vertices as follows. Rename and rearrange $\{n_1, n_2, \ldots, n_k\}$ into $\{n'_1, n'_2, \ldots, n'_k\}$ such that $n'_1 \le n'_2 \le \cdots \le n'_\ell$ are odd and $n'_{\ell+1} \le n'_{\ell+2} \le \cdots \le n'_k$ are even. Label the vertices of $C_{n'_i}$ according to

$$c'_{ij} = d'_{ij} = 1 \quad \text{if } 1 \le j < b'_i,$$

$$c'_{ib_i} = \begin{cases} \lfloor (n'_i - 2(b'_i - 1))/2 \rfloor & \text{if } i \text{ is odd,} \\ \lceil (n'_i - 2(b'_i - 1))/2 \rceil & \text{if } i \text{ is even,} \end{cases}$$

$$d'_{ib_i} = \begin{cases} \lceil (n'_i - 2(b'_i - 1))/2 \rceil & \text{if } i \text{ is odd,} \\ \lfloor (n'_i - 2(b'_i - 1))/2 \rfloor & \text{if } i \text{ is even.} \end{cases}$$

This gives a friendly labeling of $C(n_1, n_2, \ldots, n_k)$.

Example 7. To obtain a friendly labeling of $C(5, 5, 8, 9, 12)$ with $b = 13$, we proceed as follows. Rename and rearrange $(n_1, n_2, n_3, n_4, n_5) = (5, 5, 8, 9, 12)$ as $(n'_1, n'_2, n'_3, n'_4, n'_5) = (5, 5, 9, 8, 12)$. Pick $(b'_1, b'_2, b'_3, b'_4, b'_5) = (1, 2, 3, 3, 4)$, and label the vertices according to

C_5	C_5	C_9	C_8	C_{12}
00111	01001	010100111	01010011	010101000111

The labeling is friendly, with $b = 1 + 2 + 3 + 3 + 4 = 13$. □

If $2 \le b < k$, we can pick $b_i = 0$ for $1 \le i \le k - b$, and $b_i = 1$ for $i \ge k - b + 1$, and label the vertices as follows. Label C_{n_i}, where $1 \le i \le k - b$, alternately with all 0-vertices and all 1-vertices. Since $n_1 \le n_2 \le \cdots \le n_k$, we find, restricting to $C(n_1, n_2, \ldots, n_{k-b})$,

$$N = v_f(1) - v_f(0) = \sum_{i=1}^{k-b} (-1)^i n_i.$$

The next lemma is easy to establish.

Lemma 5. *Let* $0 < x_1 \le x_2 \le \cdots \le x_k$ *be a nondecreasing sequence of positive real numbers. Define* $s_t = \sum_{i=1}^{t} (-1)^i x_i$. *Then* $|s_t| \le x_t$ *if* t *is odd, and* $|s_t| < x_t$ *if* t *is even.*

Proof. The proof is based on the observation that $s_t \le 0$ if t is odd, $s_t \ge 0$ if t is even, and can be finished by induction. □

Lemma 5 gives $|N| \le n_{k-b+1} \le n_{k-b+2} \le \cdots \le n_k$. Write $|N| = bq + r$, where $0 \le r < b$. For $k - b + 1 \le i \le k - b + r$, label $q + 1$ consecutive vertices in each C_{n_i} with 0 or 1, depending on whether N is positive or negative, respectively. They will be part of the 0-vertices in the first block (or part of the 1-vertices in the last block, respectively) of C_{n_i}. For $k - b + r < i \le k$, label q consecutive vertices with 0 (or 1 respectively). This process in effect distributes $|N|$ vertices among the C_i's, where $i \ge k - b + 1$, as evenly as possible, so that the partially completed vertex labeling has $v_f(1) = v_f(0)$. The remaining vertices can now be labeled in the same manner as before. More precisely, define

$$n_i'' = \begin{cases} n_i - q - 1 & \text{if } k - b + 1 \le i \le k - b + r, \\ n_i - q & \text{if } k - b + r < i \le k. \end{cases}$$

Rename and rearrange these n_i'''s into odd numbers $m_1 \le m_2 \le \cdots \le m_\ell$ and even numbers $m_{\ell+1} \le m_{\ell+2} \le \cdots \le m_b$. Either choose

$$c_{i1} = \begin{cases} \lfloor m_i/2 \rfloor & \text{if } i \text{ is odd} \\ \lceil m_i/2 \rceil & \text{if } i \text{ is even} \end{cases} \quad \text{and} \quad d_{i1} = \begin{cases} \lceil m_i/2 \rceil & \text{if } i \text{ is odd} \\ \lfloor m_i/2 \rfloor & \text{if } i \text{ is even} \end{cases} \tag{1}$$

or

$$c_{i1} = \begin{cases} \lceil m_i/2 \rceil & \text{if } i \text{ is odd} \\ \lfloor m_i/2 \rfloor & \text{if } i \text{ is even} \end{cases} \quad \text{and} \quad d_{i1} = \begin{cases} \lfloor m_i/2 \rfloor & \text{if } i \text{ is odd} \\ \lceil m_i/2 \rceil & \text{if } i \text{ is even} \end{cases} \tag{2}$$

to ensure that we have a friendly labeling with the required b. We need two alternatives because it is possible to have $m_i = 1$, which may force some b_i to become zero.

Example 8. Consider $C(3, 4, 5, 5, 5, 8)$ and $b = 3$. We use the following steps

	C_3	C_4	C_5	C_5	C_5	C_8
Stage 1:	000	1111	00000	??11	???1	???????1
Stage 2:	000	1111	00000	01111	00111	00001111

to obtain a friendly labeling. □

Example 9. To obtain a friendly labeling of $C(3, 6, 6, 7, 7, 7, 8)$ with $b = 5$, we use the following 2-stage process

	C_3	C_6	C_6	C_7	C_7	C_7	C_8
Stage 1:	000	111111	0?????	0??????	0??????	???????	????????
Stage 2:	000	111111	000111	0000111	0000111	0000111	00001111

to label the vertices. □

Example 10. Consider $C(3, 3, 3, 3, 4)$. Assume we want $b = 2$. The initial partial labeling yields

C_3	C_3	C_3	C_3	C_4
000	111	000	?11	???1

If we use (1) to label the remaining vertices, we would end up with $b_4 = 0$ and $b_5 = 1$; hence $b = 1$. Using (2), however, yields

C_3	C_3	C_3	C_3	C_4
000	111	000	011	0111

which is a friendly labeling with $b = 2$. □

Thus far, we have shown that it is always possible to express b as a sum of the b_i's whenever $b \geq 2$. We still have to examine the possibility of writing $b = 0$ or $b = 1$ in the form of $\sum_{i=1}^{k} b_i$ for some combinations of b_i's. Clearly, $b = 0$ if and only if $b_i = 0$ for each i. The labeling will be friendly if and only if we can partition $\{n_1, n_2, \ldots, n_k\}$ into two subsets X and Y such that $|\sum_{n_i \in X} n_i - \sum_{n_j \in Y} n_j| \leq 1$.

Now we focus our attention to $b = 1$. If $|\sum_{i=1}^{k-1}(-1)^i n_i| < n_k$, we can apply the same strategy we used above to obtain $b = 1$.

Example 11. To obtain $b = 1$ in $C(3, 4, 4, 4)$, we start with the initial partial labeling

C_3	C_4	C_4	C_4
000	1111	0000	?111

Next, use (2) to complete the labeling:

C_3	C_4	C_4	C_4
000	1111	0000	0111

The result is a friendly labeling with $b = 1$. □

This labeling method is always possible if k is odd, because, according to Lemma 5, $|\sum_{i=1}^{k-1}(-1)^i n_i| < n_{k-1} \leq n_k$. We may have a problem when k is even and $|\sum_{i=1}^{k-1}(-1)^i n_i| = n_{k-1} = n_k$, as in the case of $C(3, 3, 4, 4)$. If this happens, our labeling method will yield $b = 0$; but we also have

$$0 = \sum_{i=1}^{k-2}(-1)^i n_i = \sum_{j=1}^{(k-2)/2}(n_{2j} - n_{2j-1}).$$

It follows that $n_{2j-1} = n_{2j}$ for $1 \leq j \leq (k-2)/2$. If $n_{2\alpha} < n_{2\alpha+1}$ for some α, where $1 \leq \alpha \leq (k-2)/2$, then we can switch $C_{n_{2\alpha}}$ with $C_{n_{2\alpha+1}}$, and label

$$C(n_1, \ldots, n_{2\alpha-1}, n_{2\alpha+1}, n_{2\alpha}, n_{2\alpha+2}, \ldots, n_k)$$

instead. Rename the new cycle lengths as m_i's (that is, let $m_{2\alpha} = n_{2\alpha+1}$, $m_{2\alpha+1} = n_{2\alpha}$, and $m_i = n_i$ if $i \neq 2\alpha, 2\alpha+1$). Then

$$\sum_{i=1}^{k-1}(-1)^i m_i = \left(\sum_{i=1}^{k-1}(-1)^i n_i\right) + 2(n_{2\alpha+1} - n_{2\alpha}) = -n_{k-1} + 2(n_{2\alpha+1} - n_{2\alpha}).$$

It follows from $0 < n_{2\alpha+1} - n_{2\alpha} < n_{k-1}$ that

$$\left|\sum_{i=1}^{k-1}(-1)^i m_i\right| < n_{k-1} = n_k = m_k;$$

thus a friendly vertex labeling of $C(m_1, m_2, \ldots, m_k)$ with $b = 1$ exists.

Example 12. If we label $C(3, 3, 4, 4)$ in the usual way, we will have $b = 0$. To obtain a friendly labeling with $b = 1$, we label the vertices in three stages. We first switch the two cycles C_3 and C_4 in the middle to obtain $C(3, 4, 3, 4)$:

	C_3	C_4	C_3	C_4
Stage 1:	???	????	???	????

Now the usual labeling method produces

	C_3	C_4	C_3	C_4
Stage 2:	000	1111	000	??11

Finally, we fill the remaining entries in the last cycle with 0's and 1's to fulfill the requirement $b_4 = 1$:

	C_3	C_4	C_3	C_4
Stage 3:	000	1111	000	0111

The result is a friendly labeling with $b = 1$. □

Example 13. Notice that $-n_{k-1}+2(n_{2\alpha+1}-n_{2\alpha})$ could be positive, as in the case of $C(3, 3, 7, 7, 7, 7)$. In such event, we fill the last cycle with 0's in Stage 2.

	C_3	C_7	C_3	C_7	C_7	C_7
Stage 1:	???	???????	???	???????	???????	???????
Stage 2:	000	1111111	000	1111111	0000000	0??????
Stage 3:	000	1111111	000	1111111	0000000	0000111

The result is again a friendly labeling with $b = 1$. □

We have seen that $b = 1$ is always attainable if k is odd, or if the n_i's are not all equal. What if k is even and the n_i's are all equal?

Lemma 6. *A friendly labeling of $C(n_1, n_2, \ldots, n_k)$ with $b = 1$ exists if and only if* (i) *k is odd, or* (ii) *the n_i's are not all equal.*

Proof. We only need to consider k is even, and $n_1 = n_2 = \cdots = n_k$, and show that in such event, it is impossible to have $b = 1$. Suppose, on the contrary, such a friendly labeling exists. Since the n_i's are constant, we may assume $b_1 = b_2 = \cdots = b_{k-1} = 0$ and $b_k = 1$. This requires, in each of the first $k - 1$ cycles, all the vertices to be labeled the same. This in turn implies that, restricted to the first $k - 1$ cycles, $|v(1) - v(0)|$ is a nonzero multiple of n_k. In particular, $|v(1) - v(0)| \geq n_k$. To maintain $b_k = 1$, at least one vertex in the last cycle must be labeled differently from the other vertices. Hence $|v(1) - v(0)| \leq n_k - 2$ in this cycle. It becomes clear that this labeling cannot be friendly. □

Lemma 6 asserts that the non-existence of $b = 1$ could only occur when k is even, and $n_1 = n_2 = \cdots = n_k$. We summarize what we have found in the next theorem.

Theorem 7. *Initially, set*

$$S = \begin{cases} \{-(n-4), -(n-8), \ldots, n - 4\lfloor \ell/2 \rfloor\} & \text{if } n \text{ is even,} \\ \{-(n-4), -(n-8), \ldots, n - 4\lfloor \ell/2 \rfloor - 2\} & \text{if } n \text{ is odd.} \end{cases}$$

Next, modify S as follows:

- *Remove $-(n - 4)$ from S if k is even and $n_1 = n_2 = \cdots = n_k$.*
- *Add $-n$ to S if there exists a partition of $\{n_1, n_2, \ldots, n_k\}$ into two subsets X and Y such that $|\sum_{n_i \in X} n_i - \sum_{n_j \in Y} n_j| \leq 1$.*

Then $FFI(C(n_1, n_2, \ldots, n_k)) = S$.

The friendly index set can now be extracted from the full friendly index by taking absolute value. The resulting friendly index set consists of even integers congruent to $n \pmod 4$ if n is even, and odd integers if n is odd. In particular, $\mathrm{FI}(C(n_1, n_2, \ldots, n_k)) \subseteq \mathrm{FI}(C_n)$.

We remarked earlier that the last $\lfloor \ell/2 \rfloor$ values of $\mathrm{FFI}(C_n)$ are omitted in $\mathrm{FFI}(C(n_1, n_2, \ldots, n_k))$. Let $x < n - 4$ be one of these $\lfloor \ell/2 \rfloor$ values. When n is even, $-x \equiv n \equiv -n \pmod 4$, hence $\mathrm{FFI}(C(n_1, n_2, \ldots, n_k))$ contains $-x$; consequently x can still be found in $\mathrm{FI}(C(n_1, n_2, \ldots, n_k))$. When n is odd, $-x \in \mathrm{FFI}(C(n_1, n_2, \ldots, n_k))$ only if $x \equiv n \pmod 4$, thus not all x's remain in $\mathrm{FI}(C(n_1, n_2, \ldots, n_k))$.

It is easy to decide whether $\mathrm{FFI}(C(n_1, n_2, \ldots, n_k))$ contains $\pm n$. Our final obstacle is to find the condition for $\mathrm{FI}(C(n_1, n_2, \ldots, n_k))$ to exclude $n - 4$. Notice that $-(n-4) \notin \mathrm{FFI}(C(n_1, n_2, \ldots, n_k))$ if k is even and $n_1 = n_2 = \cdots = n_k$; in which case n must be even. Meanwhile, $n - 4 \in \mathrm{FFI}(C(n_1, n_2, \ldots, n_k))$ if and only if n is even and $\ell \in \{0, 2\}$. Therefore $n - 4 \notin \mathrm{FI}(C(n_1, n_2, \ldots, n_k))$ if $n_1 = n_2 = \cdots = n_k$, k is even, and $\ell > 2$, which in turn implies that $\ell = k > 2$.

We have obtained a complete solution of our main problem.

Theorem 8. *Initially, if n is even, let*

$$T = \begin{cases} \{0, 4, 8, \ldots, n-4\} & \text{if } n \equiv 0 \pmod 4, \\ \{2, 6, 10, \ldots, n-4\} & \text{if } n \equiv 2 \pmod 4; \end{cases}$$

if n is odd, let

$$T = \begin{cases} \{1, 3, 5, \ldots, n-2\} & \text{if } \ell = 1, \\ \{1, 3, 5, \ldots, n - 4\lfloor \ell/2 \rfloor\} & \\ \quad \cup \{n - 4\lfloor \ell/2 \rfloor + 4, n - 4\lfloor \ell/2 \rfloor + 8, \ldots, n-4\} & \text{if } \ell \geq 3. \end{cases}$$

Next, modify T as follows:

- *Remove $n - 4$ from T if k is even, $k > 2$, and $n_1 = n_2 = \cdots = n_k$ are odd.*
- *Add n to T if* (i) *there exists a partition of $\{n_1, n_2, \ldots, n_k\}$ into two subsets X and Y such that $|\sum_{n_i \in X} n_i - \sum_{n_j \in Y} n_j| \leq 1$, or* (ii) *$\ell = 0$.*

Then $FI(C(n_1, n_2, \ldots, n_k)) = T$.

Example 14. Let us apply Theorem 8 to $k = 2$. We find that $\mathrm{FI}(C(n_1, n_2))$ always contains $n - 4$; and $\mathrm{FI}(C(n_1, n_2))$ contains n if either (i) $|n_1 - n_2| \leq 1$ or (ii) $\ell = 0$, which means both n_1 and n_2 are even. Therefore the result agrees with Theorem 3. Interestingly, Theorem 3 is restricted to $k = 2$, but Theorem 8 allows $k = 1$, in which event we obtain Theorem 2. □

Example 15. These results

- $\mathrm{FI}(C(8, 8, 8)) = \{0, 4, 8, \ldots, 24\}$
- $\mathrm{FI}(C(8, 8, 10)) = \{2, 6, 10, \ldots, 26\}$
- $\mathrm{FI}(C(3, 3, 3, 3)) = \{0, 4\} \cup \{12\}$
- $\mathrm{FI}(C(3, 3, 3, 4)) = \{1, 3, 5, 7, 9\} \cup \{13\}$
- $\mathrm{FI}(C(3, 3, 4, 4)) = \{2, 6, 10, 14\}$
- $\mathrm{FI}(C(3, 4, 4, 4)) = \{1, 3, 5, \ldots, 15\}$
- $\mathrm{FI}(C(3, 4, 4, 8)) = \{1, 3, 5, \ldots, 17\}$
- $\mathrm{FI}(C(6, 6, 6, 6)) = \{0, 4, 8, \ldots, 24\}$
- $\mathrm{FI}(C(5, 5, 5, 7, 9)) = \{1, 3, 5, \ldots, 23\} \cup \{27, 31\}$
- $\mathrm{FI}(C(3, 3, 4, 4, 5, 9)) = \{0, 4, 8, \ldots, 28\}$
- $\mathrm{FI}(C(3, 3, 4, 4, 7, 11)) = \{0, 4, 8, \ldots, 28\}$
- $\mathrm{FI}(C(3, 5, 7, 7, 7, 9, 9)) = \{1, 3, 5, \ldots, 35\} \cup \{39, 43, 47\}$

follow directly from Theorem 8. □

Corollary 9. *The 2-regular graph $C(n_1, n_2, \ldots, n_k)$ is cordial if and only if $n_1 + n_2 + \cdots + n_k \not\equiv 2 \pmod 4$.*

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