On Balancedness of Some Graph Constructions

Suh-Ryung Kim
Department of Mathematics Education
Seoul National University
Seoul 151-748 , Korea

Sin-Min Lee
Department of Computer Science
San Jose State University
San Jose, California 95192 U.S.A.

Ho Kuen Ng
Department of Mathematics
San Jose State University
San Jose, California 95192 U.S.A.

Abstract Let G be a graph with vertex set V(G) and edge set E(G), and let A = {0, 1}. A labeling f : V(G) → A induces a partial edge labeling f^* : E(G) → A defined by $f^*(xy) = f(x)$, if and only if f(x) = f(y), for each edge xy ∈ E(G). For i ∈ A, let $v_f(i)$ = card{v ∈ V(G) : f(v) = i} and $e_{f^*}(i)$ = card{e ∈ E(G) : $f^*(e) = i$}. A labeling f of a graph G is said to be friendly if $| v_f(0) - v_f(1) | \leq 1$. If $| e_{f^*}(0) - e_{f^*}(1) | \leq 1$ then G is said to be ***balanced***. Balancedness of the Cartesian product and composition of graphs is studied in [19]. We provide some new families of balanced graphs using other constructions.

1. Introduction.

A graph on n vertices is said to be **cordial** if there exists a labeling f of the vertex set using zeros and ones that are as equal as possible in number, so that the numbers of induced edge labels that are zero and one differ by at most one. Here the edge label induced by vertices u and v is defined by | f(u) – f(v) |.

The concept of cordial graph labeling was introduced by Cahit [3] in 1986. Cahit proved the following: every tree is cordial; K_n is cordial if and only if n ≤ 3; $K_{m,n}$ is cordial for all m and n; the wheel W_n is cordial if and only if n ≢ 3 (mod 4); C_n is cordial if and only if n ≢ 2 (mod 4); and an Eulerian graph is not cordial if its size is 2 (mod 4). Several constructions of cordial graphs, in particular, the Cartesian product, composition and tensor product are considered

JCMCC **66** (2008), pp. 3-16

in [1, 3, 4, 5,10, 11, 15, 18, 22]. Kircherr considered more general constructions of cordial graphs in [16]. Other studies can be found in [13, 17, 18, 23].

Cairnie and Edwards [6] have determined the computational complexity of cordial labeling. They proved that to decide whether a graph admits a cordial labeling is NP-complete. Even the restricted problem of deciding whether a connected graph of diameter 2 has a cordial labeling is NP-complete. For more known results and open problems on cordial graphs, see [5, 7].

Liu, Tan and the second author [19] considered a new labeling problem of graph theory. Let G be a graph with vertex set V(G) and edge set E(G). A vertex labeling of G is a mapping f from V(G) into the set {0, 1}. For each vertex labeling f of G, we can define a partial edge labeling f* of G in the following way. For each edge {u, v} in E(G), we define

$$f^*(u,v) = \begin{cases} 0 & \text{if } f(u) = f(v) = 0, \\ 1 & \text{if } f(u) = f(v) = 1. \end{cases}$$

Note that if $f(u) \neq f(v)$, then the edge {u, v} is not labeled by f*. Thus f* is a partial function from E(G) into the set {0, 1}, and we refer to f* as the induced partial function of f. Let $v_f(0)$ and $v_f(1)$ denote the number of vertices of G that are labeled by 0 and 1 under the mapping f respectively. Likewise, let $e_{f^*}(0)$ and $e_{f^*}(1)$ denote the number of edges of G that are labeled by 0 and 1 under the induced partial function f* respectively. With these notations, we now introduce the notion of a balanced graph.

Definition. Let G be a graph. G is said to be a **balanced** graph or **balanced** if there is a vertex labeling f of G such that $|v_f(0) - v_f(1)| \leq 1$ and $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$.

A graph G is said to be **strongly vertex-balanced** if G is a balanced graph and $v_f(0) = v_f(1)$. Similarly a graph G is **strongly edge-balanced** if it is a balanced graph and $e_{f^*}(0) = e_{f^*}(1)$. If G is a strongly vertex-balanced and strongly edge-balanced graph, then we say that G is a **strongly balanced** graph.

We will drop the subscripts f and f* when the context is clear.

Example 1. Figure 1 shows a graph with two distinct balanced labelings.

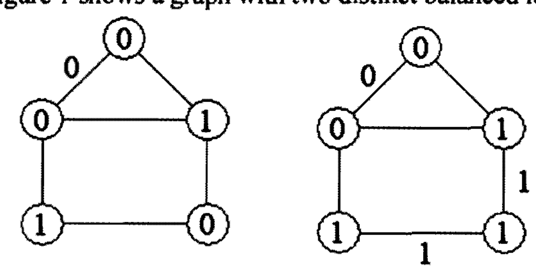
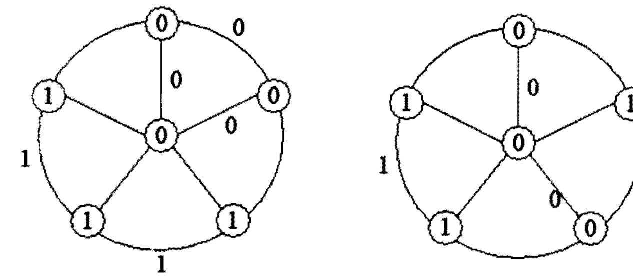


Figure 1.

Example 2. Figure 2 shows that the wheel W_5, up to isomorphism, has two balanced labelings. They are strongly vertex-balanced but not strongly edge-balanced.



W_5 $|e(1)\text{-}e(0)| = 1$ $|e(1)\text{-}e(0)| = 1$

Figure 2.

Example 3. Figure 3 depicts a strongly balanced graph.

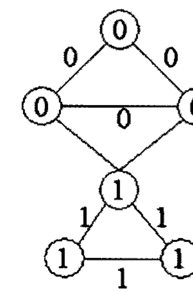


Figure 3.

The following graphs are studied in [19]:

(1) The path P_n is balanced; it is strongly balanced if n is even.
(2) The cycle C_n is balanced; it is strongly balanced if n is even.
(3) The complete graph K_n is strongly balanced if and only if n is even.
(4) The complete bipartite graph $K_{m,n}$ is balanced if and only if one of the following conditions holds,
 (a) m, n are even,
 (b) m, n are odd and $|m - n| \leq 2$,
 (c) one of m and n, say m, is odd, n = 2t and $t \equiv -1$, 0 or 1 mod ($|m - n|$).
(5) If G is k-regular with p vertices, then
 (a) G is balanced if p is odd and k = 2;
 (b) G is strongly balanced if p is even.

Though the concept of balanced graph labeling is similar to that of cordial graph labeling, the two theories are completely different. In this paper, we use different product constructions to build up significant classes of strongly balanced graphs. We show that the balancedness is not well-behaved under these graph operations.

2. Balancedness of mK_n.

Kuo et al [17] investigated the cordiality of the disjoint union of m copies of the complete graph K_n. In this section we consider the balanced problem for mK_n.

First of all, we establish the following general results.

Theorem 2.1. For any graph G, the graph mG is strongly balanced for all even m.

Proof. Label the vertices of the first component of mG by 0, those of the second component by 1, and so on alternately for the rest of the graph. It is clear that this labeling is strongly balanced.

Theorem 2.2. If G is strongly balanced and H is balanced, then the disjoint union $G \cup H$ is balanced.

Theorem 2.3. If G is balanced, then mG is balanced for all $m \geq 1$.

Notation. Suppose G is a balanced graph under the labeling f, and $v_f(0) - v_f(1) = i$, $e_{f^*}(0) - e_{f^*}(1) = j$. We say that G is (i, j)-balanced.

Lemma 2.4. If G is (i, j)-balanced under f, then the mapping $f^\#$ defined by $f^\#(u) = 1 - f(u)$ gives a $(-i, -j)$-balanced labeling for G.

We have the following result

Theorem 2.5. If G is (1, 1)-balanced and H is $(-1, -1)$-balanced, then $G \cup H$ is strongly balanced.

Theorem 2.6. The graph mK_3 is balanced for all $m > 0$, and is strongly balanced if and only if m is even.

Proof. Note that K_3 is (1, 1)-balanced with vertex labeling {0, 0, 1} and $(-1, -1)$-balanced with vertex labeling {1, 1, 0}. The results follow from Theorems 2.2 and 2.5.

However the situation is quite different for mK_n when $n \geq 4$.

Theorem 2.7. The graph mK_n is not balanced for any odd $m > 0$ and odd $n \geq 4$.

Proof. For simplicity we denote the i-th copy of mK_n by G_i.

Let f be a labeling of mK_n, and let x_i, where $0 \leq x_i \leq n$, be the number of 0-vertices in G_i. Restricting f to this G_i, we have $v(0) = x_i$, $v(1) = n - x_i$, and $e(0) = C(x_i, 2)$, $e(1) = C(n - x_i, 2)$.

Let $m = 2t + 1$, $n = 2s + 1$, and $x_1 + x_2 + \ldots + x_m = \lceil mn/2 \rceil$.

As $C(x_i, 2) - C(n - x_i, 2) = -(2s^2 + s - 2sx_i)$, $e(0) - e(1) = \Sigma(C(x_i, 2) - C(n - x_i, 2)) = -s = -(n - 1)/2$.

Thus if $n > 3$, mK_n cannot be balanced.□

Summarizing the above results, we conclude that

Theorem 2.8. The graph mK_n is balanced if and only if

(a) $n = 3$ and $m > 0$, or

(b) mn is even.

3. Balancedness of the Cartesian product of graphs.

The **Cartesian product** $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, and edge set $\{\{(a, x), (b, x)\} : \{a, b\} \in E(G)\} \cup \{\{(a, x), (a, y)\} : \{x, y\} \in E(H)\}$.

For each vertex of G, we will call the corresponding copy of H its cross section, or a G-cross section. Similarly, for each vertex of H, we will call the corresponding copy of G its cross section, or an H-cross section.

Theorem 3.1. Let H be a strongly balanced graph, and G be any graph. Then $G \times H$ is strongly balanced.

Proof. Consider a strongly balanced labeling of H. For each vertex of G, use this labeling for its cross section. Thus each G-cross section has $v(0) = v(1)$ and $e(0) = e(1)$. Then obviously $v(0) = v(1)$ for the Cartesian product. For each vertex of H, if it is labeled 0, all the vertices in its cross section are labeled 0, and if it is labeled 1, all the vertices in its cross section are labeled 1. Thus all the edges in this H-cross section are labeled 0 or 1 accordingly. Then the combined values of e(0) and the combined values of e(1) are the same when all the H-cross sections are taken together.□

Remark. The Cartesian product does not preserve balancedness. For example, C_3 is balanced. However, $C_3 \times C_3$ is not balanced. Up to isomorphism $C_3 \times C_3$ has the following three friendly labelings. However, they are not balanced. (Figure 4)

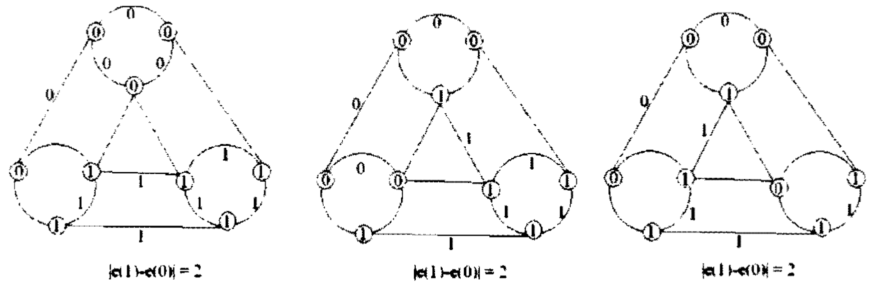


Figure 4.

Corollary 3.2. For any graph G and any positive even n, $G \times P_n$ is strongly balanced.

Corollary 3.3. For any graph G and any positive even n, $G \times C_n$ is strongly balanced.

Corollary 3.4. For any graph G and any positive even n, $G \times K_n$ is strongly balanced.

Corollary 3.5. Every graph is an induced subgraph of a strongly balanced graph.

The Cartesian product of two paths is frequently called a grid graph. These graphs are very nicely balanced.

Theorem 3.6.

(1) If one of m or n is even, $P_m \times P_n$ is strongly balanced.

(2) If both m and n are odd, $P_m \times P_n$ is balanced, strongly edge-balanced but not strongly vertex-balanced.

Proof. (1) follows from Corollary 3.2.

For (2), label the vertices alternately by 0's and 1's. Then $e(0) = e(1) = 0$, and $|v(0) - v(1)| = 1$.□

4. Balancedness of the composition of graphs.

Given two graphs G and H, the composition of G and H, denoted by G[H], is the graph with vertex set $V(G) \times V(H)$, with (u_1, v_1) adjacent to (u_2, v_2) whenever $\{u_1, u_2\} \in E(G)$ or ($u_1 = u_2$ and $\{v_1, v_2\} \in E(H)$). It is also called the graph lexicographic product.

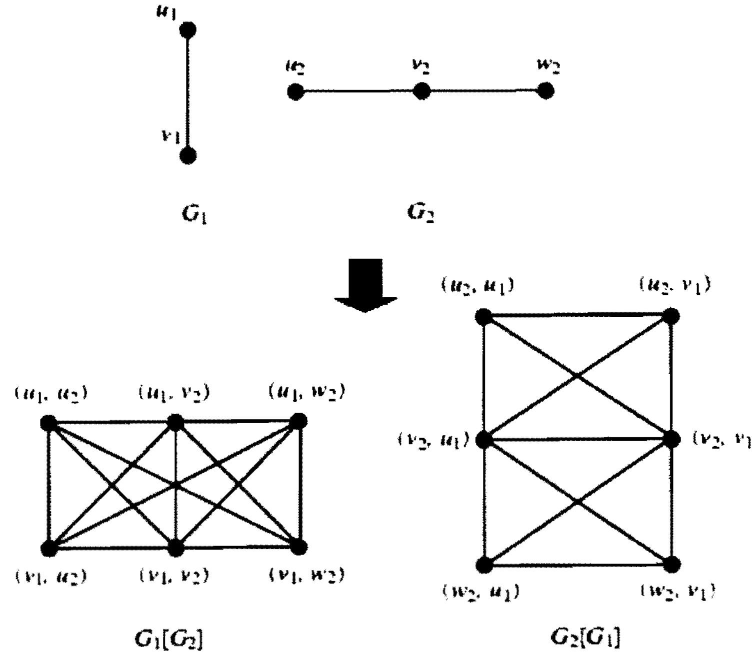


Figure 5.

It is shown in [19] that

Theorem 4.1. For any graph G and strongly balanced graph H, the composition G[H] is strongly balanced.

Corollary 4.2. For any graph G and any positive even n, $G[P_n]$ is strongly balanced.

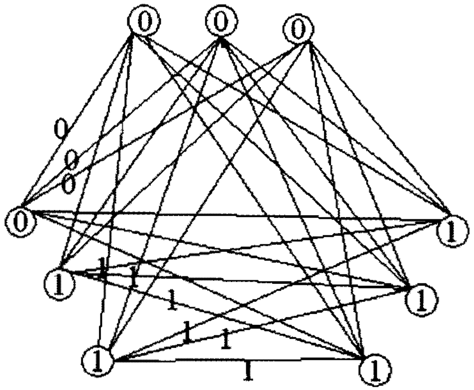
Corollary 4.3. For any graph G and any positive even n, $G[C_n]$ is strongly balanced.

Corollary 4.4. For any graph G and any positive even n, $G[K_n]$ is strongly balanced.

Remark. Theorem 4.1. is not true if "strongly balanced" is replaced with "balanced". By a result of [19], a k-regular graph with an odd number of vertices is balanced if and only if k = 2. Though K_3 is balanced, $K_3[K_3]$ is 8-regular of odd order and it is not balanced.

Corollary 4.5. For any graph G and any positive even n, the composition of G with the null graph N_n, i.e., $G[N_n]$, is strongly balanced.

Remark. The above result is not true if n is odd. The graph $K_3[N_3]$ has order 9, and is regular with degree 6. Any friendly labeling will give $|e(1) - e(0)| = 3$.



$|e(1)\text{-}e(0)| = 3$

Figure 6.

Theorem 4.6.
(1) If one of m or n is even, then $P_m[P_n]$ is strongly balanced.
(2) If both m and n are odd, $P_m[P_n]$ is balanced, strongly edge-balanced but not strongly vertex-balanced.

Example 4. Figure 7 shows that $P_3[P_3]$ is balanced, strongly edge-balanced but not strongly vertex-balanced.

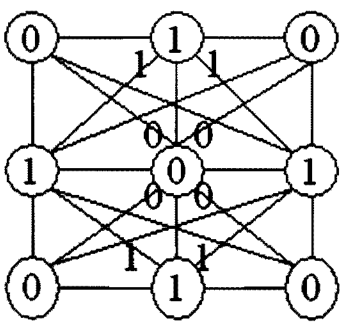


Figure 7.

5. Balancedness of the tensor product of graphs.

The **tensor product** $G = G_1 \otimes G_2$ of graphs G_1 and G_2 with disjoint point sets V_1 and V_2 and edge sets E_1 and E_2 is the graph with vertex set $V_1 \times V_2$ such that (u_1, u_2) is adjacent to (v_1, v_2) whenever $\{u_1, v_1\} \in E_1$ and $\{u_2, v_2\} \in E_2$. We see that if G_1 is (p_1, q_1)-graph and G_2 is (p_2, q_2)-graph, then $G_1 \otimes G_2$ is a $(p_1p_2, 2q_1q_2)$-graph. The construction was originally introduced by Weischel [26]. It is also called the Kronecker product, weak product, direct product, categorical product and conjuction in the literature. The tensor product of graphs was considered in Borowicki [2], Culik [9], Miller [21], and Weichel [26] under different terminologies.

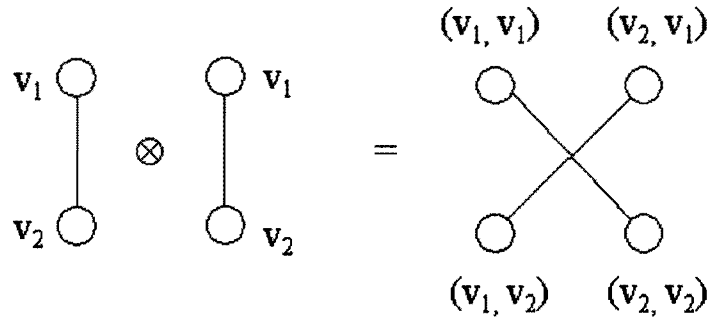
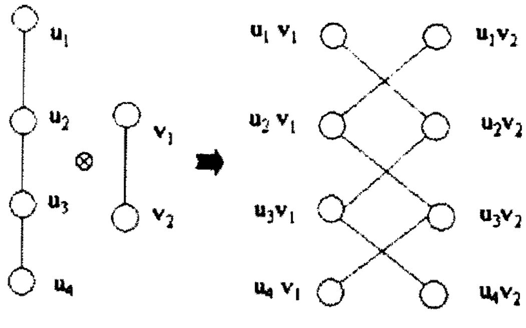


Figure 8.

Weichel [26] showed that the tensor product of two connected graphs is connected if and only if one of the factors has an odd cycle. For other results of tensor products, the reader is referred to [2, 7, 9].

Theorem 5.1. If G is bipartite, then $G \otimes K_2$ is strongly balanced.
Proof. $G \otimes K_2 = 2G$, by Weichel's result. Thus the graph is strongly balanced by labeling one component of 2G by 0 and the other by 1.□

Example 5. $P_4 \otimes K_2$ is strongly balanced.



$P_4 \otimes P_2$

Figure 9.

The tensor product of two balanced graphs can be balanced. For example the graph $P_3 \otimes P_3$ is balanced. However, there exist tensor products with balanced components that are themselves not balanced. As an example, consider the graph $K_3 \otimes K_3$. The graph is 4-regular with 9 vertices. By a result in [19], a k-regular graph with an odd number of vertices is balanced if and only if $k = 2$. We therefore conclude that $K_3 \otimes K_3$ is not balanced.

For a function $f: X \to X$ and a subset A of X, define the inverse image of A as the set $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$.

Consider graphs G and H, and their tensor product $G \otimes H$. We define the projection functions π_G: $V(G \otimes H) \to V(G)$ and π_H: $V(G \otimes H) \to V(H)$ by

$\pi_G(x, y) = x$ and $\pi_H(x, y) = y$.

Lemma 5.2.
(i) $\pi_G^{-1}(\{x_i\}) \cap \pi_G^{-1}(\{x_k\}) = \emptyset$ if $i \neq k$ and $\pi_H^{-1}(\{y_j\}) \cap \pi_H^{-1}(\{y_l\}) = \emptyset$ if $j \neq l$.
(ii) $|\pi_G^{-1}(\{x_i\})| = |V(H)|$ and $|\pi_H^{-1}(\{y_j\})| = |V(G)|$.
Proof.
(i) Suppose $\alpha \in \pi_G^{-1}(\{x_i\}) \cap \pi_G^{-1}(\{x_k\})$. Then $\alpha = (x_i, y_j) = (x_k, y_l)$ for some y_j and $y_l \in V(H)$. By the property of ordered pairs, $x_i = x_k$, contradicting the hypothesis. A similar argument applies for the second statement.
(ii) $|\pi_G^{-1}(\{x_i\})| = |\{(x_i, y_j) \mid y_j \in H\}| = |V(H)|$,
and $|\pi_H^{-1}(\{y_j\})| = |\{(x_i, y_j) \mid x_i \in G\}| = |V(G)|$.□

Lemma 5.3.
(i) For distinct edges $\{y_j, y_l\}$ and $\{y_p, y_q\}$ of H,
$\{\{(x_i, y_j), (x_k, y_l)\} \in G \otimes H \mid \{x_i, x_k\} \in E(G)\}$ and
$\{\{(x_i, y_p), (x_k, y_q)\} \in G \otimes H \mid \{x_i, x_k\} \in E(G)\}$ are disjoint.

(ii) For distinct edges $\{x_i, x_k\}$ and $\{x_p, x_q\}$ of G,
$\{\{(x_i, y_j), (x_k, y_l)\} \in G \otimes H \mid (y_j, y_l) \in E(H)\}$ and
$\{\{(x_p, y_j), (x_q, y_l)\} \in G \otimes H \mid (y_j, y_l) \in E(H)\}$ are disjoint.
Proof. It immediately follows from the property of ordered pairs that $(u, v) \neq (r, s)$ if and only if $u \neq r$ or $v \neq s$.

For graphs G and H, and their tensor product $G \otimes H$, define functions p_G: $E(G \otimes H) \to E(G)$ and p_H: $E(G \otimes H) \to E(H)$ as follows:

$p_G(\{(x_i, y_j), (x_k, y_l)\}) = \{x_i, x_k\}$ and $p_H(\{(x_i, y_j), (x_k, y_l)\}) = \{y_j, y_l\}$.

Lemma 5.4.
(i) For any edge $\{x_i, x_k\}$ of G, $|p_G^{-1}(\{\{x_i, x_k\}\})| = |E(H)|$.
(ii) For any edge $\{y_j, y_l\}$ of H, $|p_H^{-1}(\{\{y_j, y_l\}\})| = |E(G)|$.
Proof. For any edge $\{x_i, x_k\}$ of G, $p_G^{-1}(\{\{x_i, x_k\}\}) = \{\{(x_i, y_j), (x_k, y_l)\} \in E(G \otimes H)\} = \{\{y_j, y_l\} \in E(H)\}$ since $\{(x_i, y_j), (x_k, y_l)\} \in E(G \otimes H)$ if and only if $\{y_j, y_l\} \in E(H)$.

A similar argument can be applied to prove the second statement.□

Lemma 5.5. Suppose that G and H are graphs. Let L: $V(H) \to \{0, 1\}$ be a labeling of H. Label a vertex (x, y) of $G \otimes H$ as m if $L(y) = m$.
(i) The number of vertices in $V(G \otimes H)$ with label m = (the number of vertices in V(H) with label m) . |V(G)|.
(ii) The number of edges in $E(G \otimes H)$ with label m = (the number of edges in E(H) with label m) . |E(G)|.
Proof.
(i) The number of vertices in $V(G \otimes H)$ with label m
$= |\{(x, y) \in V(G \otimes H) \mid L(y) = m\}|$
$= \sum |\pi_H^{-1}(\{y\}) : L(y) = m|$ since sets of the form $\pi_H^{-1}(\{y\})$ are pairwise disjoint, by Lemma 5.2(i)
= (the number of vertices in V(H) with label m) . |V(G)|, by Lemma 5.2(ii).
(ii) Let S be the set of edges in $G \otimes H$ with label m and T be the set $E(G) \times M$ where M is the set of edges in H with label m. Define a map ϕ from S to T by

$$\phi(\{(x_i, y_k), (x_j, y_l)\}) = (\{x_i, x_j\}, \{y_k, y_l\}).$$

We will show that ϕ is a bijection. Take an edge $\{(x_i, y_k), (x_j, y_l)\}$ in $G \otimes H$ with label m. Then $\{x_i, x_j\} \in E(G)$ and $\{y_k, y_l\} \in E(H)$ by the definition of tensor product. In addition, both (x_i, y_k) and (x_j, y_l) have label m by the definition of partial edge labeling. By the way how the labeling on $V(G \otimes H)$ is defined, y_k and y_l both have label m in H. Then edge $\{y_k, y_l\}$ has label m in H. Thus ϕ is well-defined.

Take edge $\{y_k, y_l\}$ in H with label m. Then vertices y_k and y_l both have label m. Now for each edge $\{x_i, x_j\}$ in G, vertices (x_i, y_k) and (x_j, y_l) have label m and are adjacent in $G \otimes H$. Moreover the edge $\{(x_i, y_k), (x_j, y_l)\}$ has label m. Thus ϕ is onto. It is easy to check that ϕ is one-to-one. Hence ϕ is a bijection.

Since $|E(G) \times M|$ = (the number of edges in E(H) with label m) $\cdot$ $|E(G)|$, the equality follows.□

Theorem 5.6. If G and H are graphs with H strongly balanced, then $G \otimes H$ is strongly balanced.

Proof. Choose a strongly balanced labeling L of H. For m = 0 or 1, label a vertex (x, y) of $G \otimes H$ with m if L(y) = m.

For the vertex labeling, the number of vertices in $V(G \otimes H)$ with label 0
= (the number of vertices in V(H) with label 0) . $|V(G)|$, by Lemma 5.5(i)
= (the number of vertices in V(H) with label 1) . $|V(G)|$, since L is a strongly balanced labeling
= the number of vertices in $V(G \otimes H)$ with label 1.

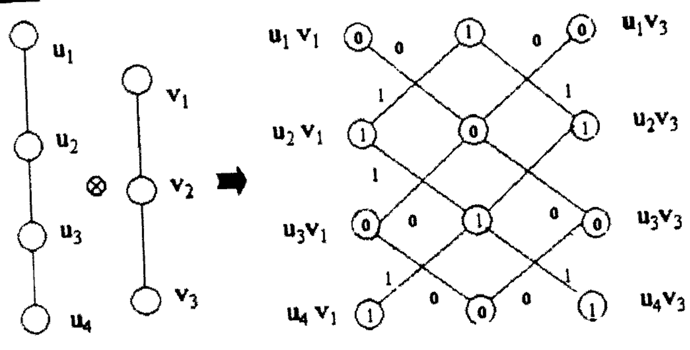
For the edge labeling, the number of edges in $E(G \otimes H)$ with label 0
= (the number of edges in E(H) with label 0) . $|E(G)|$, by Lemma 5.5(ii)
= (the number of edges in E(H) with label 1) . $|E(G)|$, since L is a strongly balanced labeling
= the number of edges in $E(G \otimes H)$ with label 1.

Hence we conclude that $G \otimes H$ is strongly balanced.□

Corollary 5.7. If G and H are graphs with G strongly balanced, then $G \otimes H$ is strongly balanced.

Proof. A similar argument can be applied.□

Example 6. P_4 is strongly balanced. Thus $P_4 \otimes P_3$ is also strongly balanced.



$P_4 \otimes P_3$

Figure 10.

Corollary 5.8. $G \otimes C_{2k}$ and $C_{2k} \otimes G$ are strongly balanced, where C_{2k} denotes a cycle with length 2k.

Corollary 5.9. $P_m \otimes P_n$ is strongly balanced if m or n is a positive even integer.

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