

On The Balance Index Sets of Graphs

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Abstract Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and let $A = \{0, 1\}$. A labeling $f: V(G) \rightarrow A$ induces a partial edge labeling $f^*: E(G) \rightarrow A$ defined by $f^*(xy) = f(x)$, if and only if $f(x) = f(y)$, for each edge $xy \in E(G)$. For $i \in A$, let $v_f(i) = \text{card}\{v \in V(G) : f(v) = i\}$ and $e_f(i) = \text{card}\{e \in E(G) : f^*(e) = i\}$. A labeling f of a graph G is said to be friendly if $|v_f(0) - v_f(1)| \leq 1$. If, $|e_f(0) - e_f(1)| \leq 1$ then G is said to be **balanced**. The **balance index set** of the graph G , $BI(G)$, is defined as $\{|e_f(0) - e_f(1)| : \text{the vertex labeling } f \text{ is friendly}\}$. Results parallel to the concept of friendly index sets are presented.

1. Introduction.

In [2], A. Liu, S.K. Tan and the second author considered a new labeling problem of graph theory. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A vertex labeling of G is a mapping f from $V(G)$ into the set $\{0, 1\}$. For each vertex labeling f of G , we can define a partial edge labeling f^* of G in the following way. For each edge $(u, v) \in E(G)$, where $u, v \in V(G)$, we have $f^*(u, v) = 1$ if $f(u) = f(v) = 1$, and $= 0$ if $f(u) = f(v) = 0$. Note that if $f(u) \neq f(v)$, the edge (u, v) is not labeled by f^* . Thus f^* is a partial function from $E(G)$ into the set $\{0, 1\}$, and we shall refer to f^* as the induced partial edge labeling.

For $i = 0, 1$, let $v_f(i) = |\{v \in V(G) \mid f(v) = i\}|$ and $e_f(i) = |\{e \in E(G) \mid f^*(e) = i\}|$. The mapping f is said to be **friendly** if $|v_f(0) - v_f(1)| = 1$

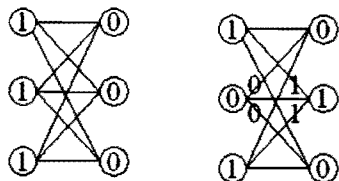
With these notations, we now introduce the notion of a balanced graph.

Definition 1. A graph G is said to be a **balanced** graph or G is **balanced** if there is a vertex labeling f of G satisfying $|v_f(0) - v_f(1)| = 1$ and $|e_f(0) - e_f(1)| = 1$.

We will use $v(0)$, $v(1)$, $e(0)$, $e(1)$ instead of the more complicated $v_f(0)$, $v_f(1)$, $e_f(0)$, $e_f(1)$, when the context is clear. A graph G is said to be **strongly vertex-balanced** if G is balanced and $v(0) = v(1)$. Similarly, a graph G is **strongly edge-balanced** if it is balanced and $e(0) = e(1)$. If G is a strongly vertex-balanced and strongly edge-balanced graph, then we say that G is a **strongly balanced** graph.

Definition 2. The **balance index set** of a graph G , $BI(G)$, is defined as $\{e_f(0) - e_f(1) : \text{the vertex labeling } f \text{ is friendly}\}$.

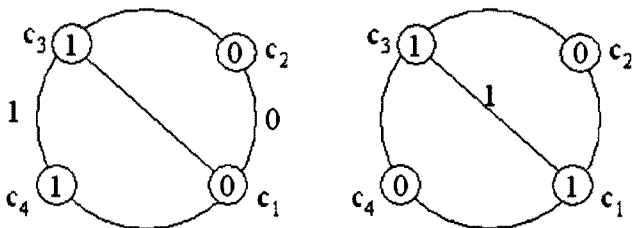
Example 1. $BI(K_{3,3}) = \{0\}$.



$$|e(1) - e(0)| = 0$$

Figure 1

Example 2. $BI(G) = \{0, 1\}$.



$$|e(1) - e(0)| = 0$$

$$|e(1) - e(0)| = 1$$

Figure 2.

Theorem 1.1. Let G be a graph with p vertices. Then $\max\{BI(G)\} = k(k-1)/2$ if $p = 2k$ or $2k-1$, depending on whether p is even or odd.

Proof. By friendliness, $v(1) = k$ if $p = 2k$ is even, and k or $k-1$ if $p = 2k-1$ is odd. Thus the maximum value of $e(1) = k(k-1)/2$. The minimum value of $e(1)$ is of course 0. The same argument gives the same maximum and minimum values for $e(0)$. The result follows.

Theorem 1.2. Let T be a tree with p vertices. Then $\max\{BI(T)\} = (p/2) - 1$ if p is even, and $(p-1)/2$ if p is odd.

Proof. By friendliness, $v(1) = p/2$ if p is even, and $(p-1)/2$ or $(p+1)/2$ if p is odd.

For a friendly vertex labeling, consider the subgraph of T containing all the edges labeled 1 and their vertices. Each connected component of this subgraph is a tree, and so the number of edges (all labeled 1) is 1 less than the number of vertices (all labeled 1). Thus $e(1) \leq v(1) - 1$ - the number of connected components $\leq v(1) - 1 = (p/2) - 1$ if p is even, and $(p-1)/2$ if p is odd. The minimum value of $e(1)$ is of course 0. The same argument gives the same maximum and minimum values for $e(0)$. The result follows.

Definition 3. A subset X of Z is said to be **BI-representable** if there exists a graph G such that $BI(G) = X$.

We investigate sets of integers that are BI-representable. Some balance graphs were considered in [6]. The notion of friendly index sets of graphs which is similar to balance index sets were considered in [3,4,5].

2. On balance index sets of some trees.

In this section we first consider balance index sets of the star $St(n)$, which is the one-point union of n copies of K_2 . We note that when computing the balance index set of a graph, we may fix an arbitrary vertex in the graph and label it 0. If another vertex labeling gives it the label 1, simply replace each vertex label by its complement. Then $v(0)$ and $v(1)$ are interchanged, and $e(0)$ and $e(1)$ are interchanged. Since we are only concerned with absolute values, interchanging $v(0)$ and $v(1)$, $e(0)$ and $e(1)$ would not make any difference.

Theorem 2.1. The balance index set of the star $St(n)$ is

- (1) $\{k\}$, if $n = 2k + 1$ is odd,
- (2) $\{k-1, k\}$, if $n = 2k$ is even.

Proof. (1) Let $n = 2k + 1$. Without loss of generality, let the center be labeled 0. Then k of the other vertices are labeled 0, while the remaining $(k+1)$ vertices are labeled 1. Thus $BI(St(2k+1)) = \{k\}$.

(2) Let $n = 2k$. Without loss of generality, let the center be labeled 0. Then either k of the other vertices are labeled 0 while the remaining k vertices are labeled 1, or $(k-1)$ of the other vertices are labeled 0 while the remaining $(k+1)$ vertices are labeled 1. Thus $BI(St(2k)) = \{k-1, k\}$. \square

Note. Theorem 2.1 shows that the maximum in Theorem 1.2 is attainable.

Corollary 2.2. For any $k > 0$, the sets $\{k\}$ and $\{k-1, k\}$ are BI-representable.

The double star $D(m, n)$ is a tree of diameter three such that there are m appended edges on one end of P_2 and n appended edges on the other end (Figure 3). Without loss of generality, we assume $m \leq n$.

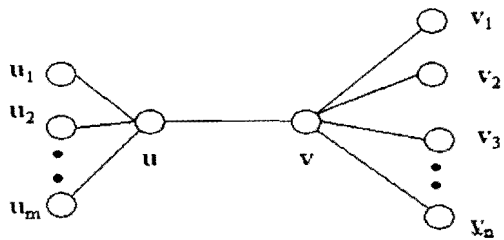


Figure 3.

Theorem 2.3. The balance index set of the double star $D(m, n)$, where $m \leq n$, is

- (1) $\{(n - m)/2, (n + m)/2\}$ if $m + n$ is even, and
- (2) $\{(n - m - 1)/2, (n - m + 1)/2, (n + m - 1)/2, (n + m + 1)/2\}$ if $m + n$ is odd.

Proof.

(1) Let $m + n = 2k$.

There are $2k + 2$ vertices. Without loss of generality, assume that the vertex u has label 0.

First label the vertex v by 0. Assume j of the vertices u_1, \dots, u_m are labeled by 0, and the other $(m - j)$ vertices are labeled by 1. By friendliness, $(k - j - 1)$ of the vertices v_1, \dots, v_n are labeled by 0, and the other $(n - k + j + 1)$ vertices are labeled by 1. Then $e(0) = k$ and $e(1) = 0$, making $e(0) - e(1) = k = (n + m)/2$.

Then label the vertex v by 1. Assume j of the vertices u_1, \dots, u_m are labeled by 0, and the other $(m - j)$ vertices are labeled by 1. By friendliness, $(k - j)$ of the vertices v_1, \dots, v_n are labeled by 0, and the other $(n - k + j)$ vertices are labeled by 1. Then $e(0) = j$ and $e(1) = n - k + j$, making $e(0) - e(1) = k - n = (m - n)/2$, with absolute value $(n - m)/2$.

Thus $BI(D(m, n)) = \{(n - m)/2, (n + m)/2\}$.

(2) Let $m + n = 2k + 1$.

There are $2k + 3$ vertices. Without loss of generality, assume that the vertex u has label 0.

First label the vertex v by 0. Assume j of the vertices u_1, \dots, u_m are labeled by 0, and the other $(m - j)$ vertices are labeled by 1. By friendliness, either $(k - 1)$ or $(k - j)$ of the vertices v_1, \dots, v_n are labeled by 0, and the other $(n - k + j + 1)$ or $(n - k + j)$ vertices are labeled by 1 respectively. Then $e(0) = k$ or $k + 1$ and $e(1) = 0$, making $e(0) - e(1) = k = (n + m - 1)/2$, or $k + 1 = (n + m + 1)/2$ respectively.

Then label the vertex v by 1. Assume j of the vertices u_1, \dots, u_m are labeled by 0, and the other $(m - j)$ vertices are labeled by 1. By friendliness, either $(k - j)$ or $(k - j + 1)$ of the vertices v_1, \dots, v_n are labeled by 0, and the other $(n - k + j)$ or $(n - k + j - 1)$ vertices are labeled by 1 respectively. Then $e(0) = j$ and $e(1) = n - k + j$ or $n - k + j - 1$, making $e(0) - e(1) = k - n = (m - n - 1)/2$, with absolute value $(n - m + 1)/2$, or $k - n + 1 = (m - n + 1)/2$, with absolute value $(n - m - 1)/2$.

Thus $BI(D(m, n)) = \{(n - m - 1)/2, (n - m + 1)/2, (n + m - 1)/2, (n + m + 1)/2\}$. \square

Corollary 2.4. For any non-negative integers a and b , with $a < b$, $\{a, b\}$ is BI-representable.

Proof. Let $m = b - a$ and $n = b + a$. Use (1) in Theorem 2.3. \square

Corollary 2.5. For any non-negative integers a and b , with $a + 1 < b$, $\{a, a + 1, b, b + 1\}$ is BI-representable.

Proof. Let $m = b - a$ and $n = b + a + 1$. Use (2) in Theorem 2.3. \square

Corollary 2.6. For any non-negative integer a , $\{a, a + 1, a + 2\}$ is BI-representable.

Proof. Let $m = 1$ and $n = 2a + 2$. Use (2) in Theorem 2.3. \square

Example 3. Figure 4 shows the balance index set of the double star $D(m, m)$, $m = 2$ and 3.

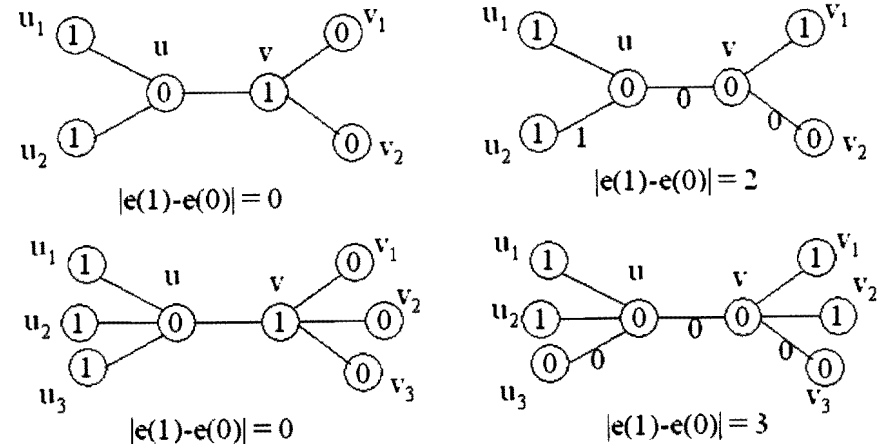


Figure 4.

Example 4. Figure 5 shows $BI(D(3, 4)) = \{0, 1, 3, 4\}$.

By Theorem 1.2, $\max\{\text{BI}(B(2, d))\} = (p-1)/2 = (2^{d+1} - 2)/2 = 2^d - 1$. This finishes the proof. \square

Example 5. $\text{BI}(B(2, 2)) = \{0, 1, 2, 3\}$.

$B(2,2)$

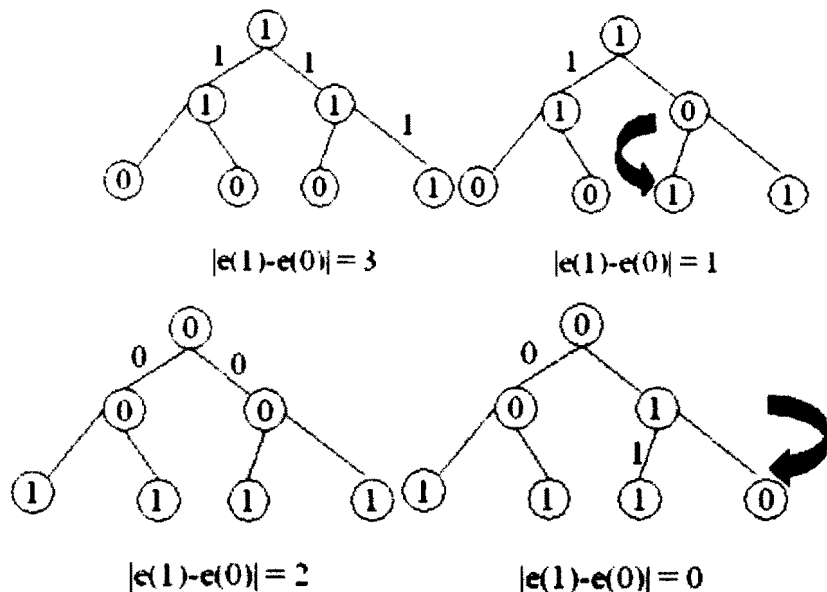


Figure 6.

Example 6. $\text{BI}(B(2, 3)) = \{0, 1, 2, 3, 4, 5, 6, 7\}$.

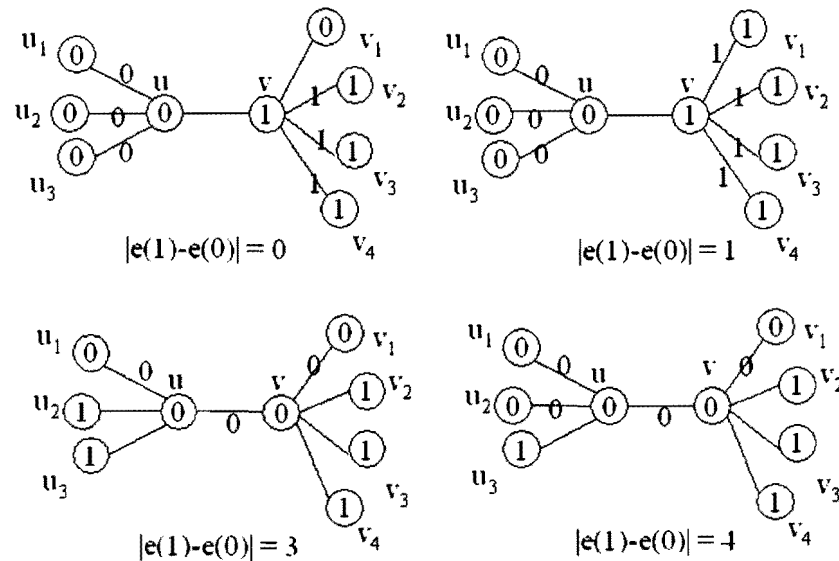


Figure 5.

Let $B(2, d)$ denote the full binary tree of depth d . We denote its vertices at depth t from left to right by $v_{1,t}, v_{2,t}, \dots, v_{2^{t-1},t}$.

Theorem 2.7. For any $d \geq 1$, the balance index set of the full binary tree $B(2, d)$ of depth d is the set $\{0, 1, 2, \dots, 2^d - 1\}$.

Proof. The tree $B(2, d)$ has $1 + 2 + 4 + \dots + 2^d = 2^{d+1} - 1$ vertices, and $2 + 4 + \dots + 2^d = 2^{d+1} - 2$ edges.

First we label all the vertices at depth less than d by 1. Then we label all the vertices at depth d except the rightmost one by 0, and the rightmost vertex by 1. We see that $e(1) = 2^d - 1$, $e(0) = 0$, and so $2^d - 1 \in \text{BI}(B(2, d))$.

At the $(d-1)$ st level, interchange the vertex labels of $v_{2^{d-1},d-1}$ and its left child $v_{2^{d-1},d}$. Then $v_{2^{d-1},d-1}$ and $v_{2^{d-1},d}$ have labels 0 and 1 respectively, and the tree has two fewer 1-edges. This decreases the value of $e(1) - e(0)$ for the tree to $(2^d - 3)$. Now interchange the vertex labels of $v_{2^{d-2},d-1}$ and its left child $v_{2^{d-2},d}$. This decreases the value of $e(1)$ by 1 and increases the value of $e(0)$ by 1, making $e(1) - e(0) = 2^d - 5$. Repeating this procedure till the vertex $v_{2^{d-1},d-1}$ produces the values $1, 3, 5, \dots, 2^d - 1$.

Now we start over, and label all the vertices at depth less than d by 1, and all the vertices at depth d by 0. We see that $e(1) = 2^d - 2$, $e(0) = 0$, and so $2^d - 2 \in \text{BI}(B(2, d))$.

At the $(d-1)$ st level, interchange the vertex labels of $v_{2^{d-1},d-1}$ and its right child $v_{2^d,d}$. Then $v_{2^{d-1},d-1}$ and $v_{2^d,d}$ have labels 0 and 1 respectively. This decreases the value of $e(1)$ by 1 and increases the value of $e(0)$ by 1, changing the value $e(1) - e(0)$ to $2^d - 4$. Repeating this procedure till the vertex $v_{2^{d-1},d-1}$ produces the values $0, 2, 4, \dots, 2^d - 2$.

Ex(2.3)

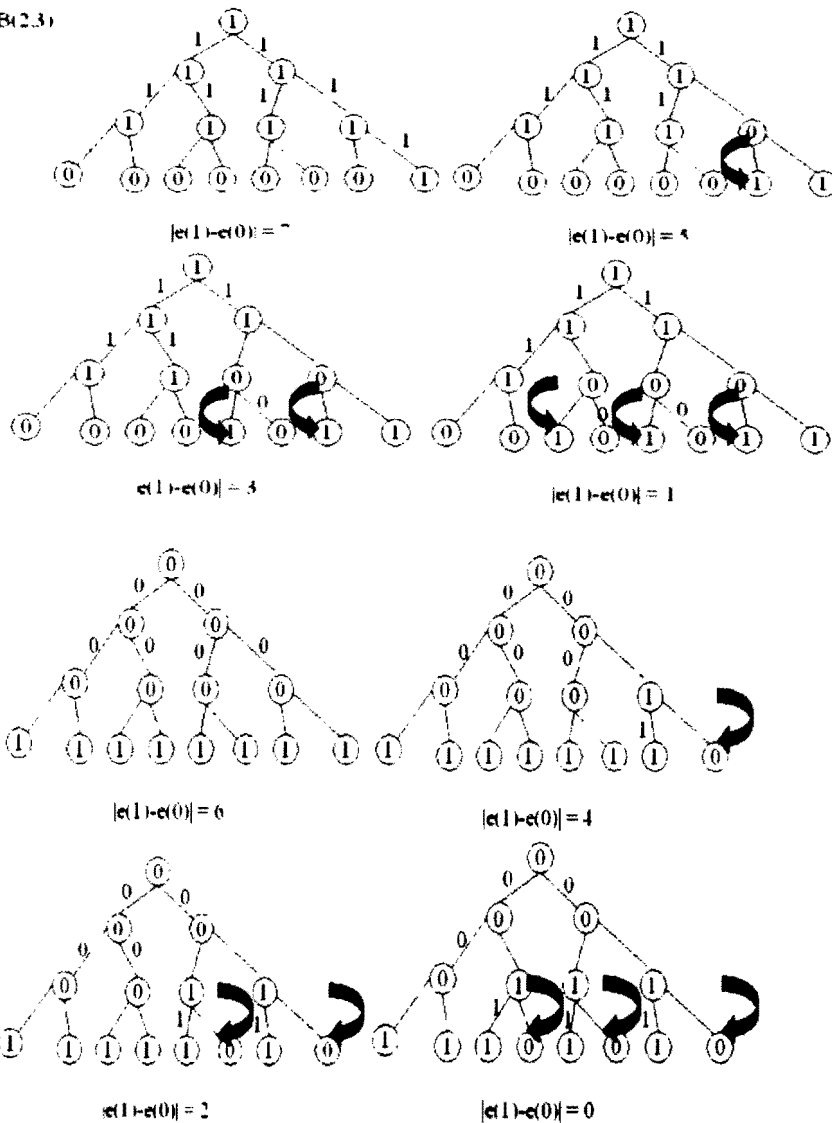


Figure 7.

3. Balance index sets of complete graphs, complete bipartite graphs and $K_n \cup N_n$.

Theorem 3.1. For any $n \geq 3$, $BI(K_n)$ is

- (1) $\{0\}$ if n is even, and
- (2) $\{k\}$ if $n = 2k + 1$ is odd.

Proof. It is clear that the balance index set of the complete graph K_{2t} is $\{0\}$ for all $t \geq 1$.

Now consider K_{2k+1} . Without loss of generality, assume that $v(1) - v(0) = 1$. Let $f(v_i) = 1$, for $i = 1, 2, \dots, k + 1$ and $f(v_j) = 0$, for $j = k + 2, \dots, 2k + 1$. Then we have $e(1) = C(k + 1, 2)$ and $e(0) = C(k, 2)$. Thus $|e(0) - e(1)| = k$. \square

For any $m, n \geq 1$, we denote the complete bipartite graph by $K_{m,n}$. Without loss of generality, we assume that $m \leq n$.

Theorem 3.2. The balance index set of complete bipartite graph $BI(K_{m,n})$ is

- (1) $\{(n - m)(i - \frac{1}{2}m) : i = 0, 1, 2, \dots, m\}$ if $(m + n)$ is even, and
- (2) $\{|(n - m)(i - \frac{1}{2}m) - \frac{1}{2}m|, |(n - m)(i - \frac{1}{2}m) + \frac{1}{2}m| : i = 0, 1, 2, \dots, m\}$ if $(m + n)$ is odd.

Proof. The set of vertices of $K_{m,n}$ can be partitioned into two subsets, one with m vertices, and the other with n vertices. Two vertices are adjacent if and only if they come from different subsets.

(1) Assume that i vertices from the first subset are labeled 0. Then the remaining $(m - i)$ vertices in this subset must be labeled 1. By friendliness, in the second subset, $(\frac{1}{2}(n + m) - i)$ vertices must be labeled 0, and the remaining $(\frac{1}{2}(n - m) + i)$ vertices must be labeled 1. It follows that $e(0) = i(\frac{1}{2}(n + m) - i)$ and $e(1) = (m - i)(\frac{1}{2}(n - m) + i)$. Simplification gives $e(0) - e(1) = (n - m)(i - \frac{1}{2}m)$. Since $i = 0, 1, 2, \dots, m$, by exhausting all the values of i , we obtain the balance index set stated in the theorem.

(2) We consider two subcases:

Case 2.1: $v(0) = \frac{1}{2}(m + n - 1)$ and $v(1) = \frac{1}{2}(m + n + 1)$.

If the first subset has i vertices labeled 0, then the remaining $(m - i)$ vertices will have label 1. Then the second subset must have $(\frac{1}{2}(m + n - 1) - i)$ vertices labeled 0, and the remaining $(\frac{1}{2}(n - m + 1) + i)$ vertices labeled 1. Then $e(0) = i(\frac{1}{2}(m + n - 1) - i)$ and $e(1) = (m - i)(\frac{1}{2}(n - m + 1) + i)$, giving $e(0) - e(1) = (n - m)(i - \frac{1}{2}m) - \frac{1}{2}m$. Since $i = 0, 1, 2, \dots, m$, by exhausting all the values of i , we obtain the first half of the set in (2).

Case 2.2: $v(0) = \frac{1}{2}(m + n + 1)$ and $v(1) = \frac{1}{2}(m + n - 1)$.

Again if the first subset has i vertices labeled 0, then the remaining $(m - i)$ vertices will be labeled 1. Then the other subset must have $(\frac{1}{2}(m + n + 1) - i)$ vertices labeled 0, and the remaining $(\frac{1}{2}(n - m - 1) + i)$ vertices labeled 1. Then $e(0) = i(\frac{1}{2}(m + n + 1) - i)$ and $e(1) = (m - i)(\frac{1}{2}(n - m - 1) + i)$, giving $e(1) - e(0) = (n - m)(i - \frac{1}{2}m) + \frac{1}{2}m$. By exhausting all the values of $i = 0, 1, 2, \dots, m$, we obtain the second half of the set in (2). \square

In particular, we have

Corollary 3.3. The balance index set of the complete bipartite graph $BI(K_{m,m})$ is $\{0\}$.

Example 7. Figure 8 illustrates $BI(K_{2,5}) = \{1, 2, 4\}$.

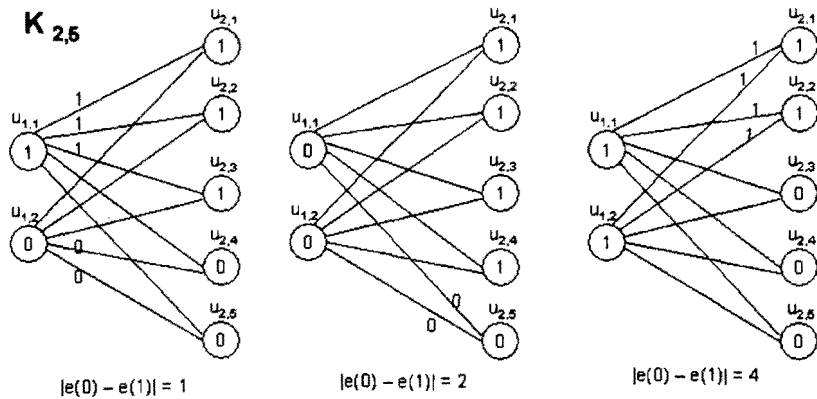


Figure 8.

The following result shows that we can generate arbitrarily large BI-representable sets.

Corollary 3.4. The balance index set of the complete bipartite graph $BI(K_{n,m+1})$ is $\{0, 1, 2, \dots, m\}$.

Example 8. Figure 9 shows the balance index sets of the complete bipartite graphs $K_{2,3}$ and $K_{3,4}$.

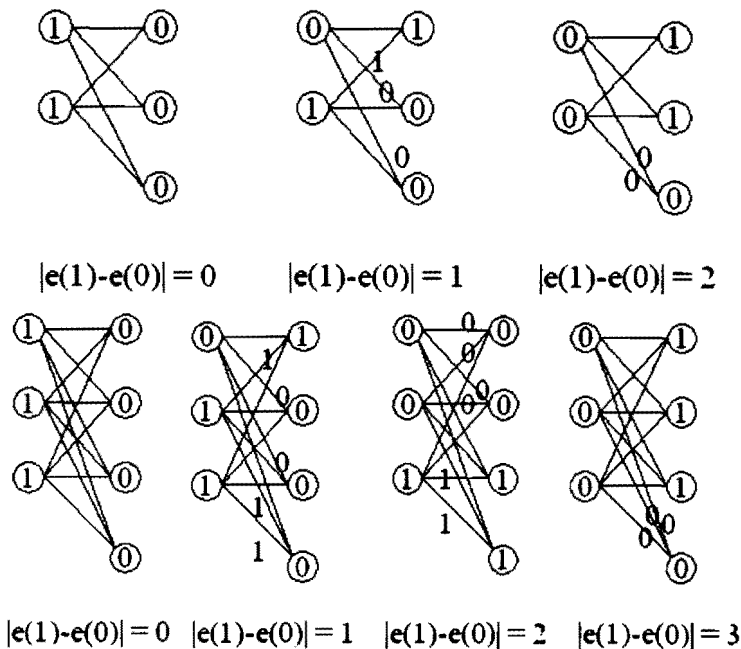
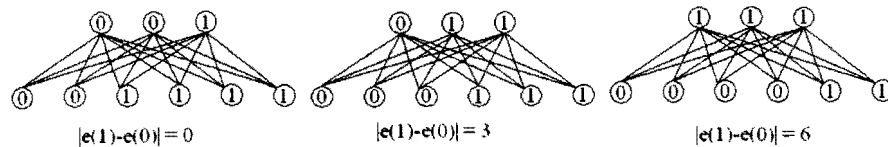


Figure 9.

Example 9. $BI(K_{3,6}) = \{0, 3, 6\}$ and $BI(K_{3,7}) = \{2, 6\}$.
 $BI(K_{3,6}) = \{0, 3, 6\}$



$BI(K_{3,7}) = \{2, 6\}$

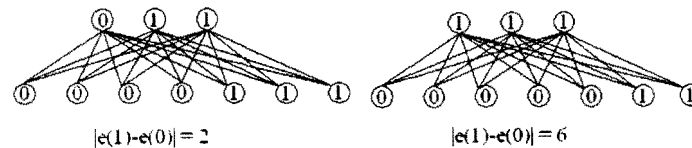


Figure 10.

Corollary 3.5. For any $n \geq 1$, $BI(K_{n,n+2}) = \{n, n-2, \dots, 0\}$ if n is even, and $\{n, n-2, \dots, 1\}$ if n is odd.

Corollary 3.6. For any $m \geq 0$, even $n \geq 2$, $BI(K_{n,n+2m}) = \{mn, m(n-2), \dots, 0\}$.

Corollary 3.7. For any odd $n \geq 1$, $\{n, n-2, \dots, 1\}$ is BI-representable.

Corollary 3.8. For any $m \geq 0$, even $n \geq 2$, $\{mn, m(n-2), \dots, 0\}$ is BI-representable.

Now we consider the disjoint union of two graphs $K_n \cup N_n$.

Theorem 3.9. For any $n \geq 2$, $BI(K_n \cup N_n) = \{\frac{1}{2}(n-1)(n-2j) : 0 \leq j \leq n/2\}$. This is $\{0, 2k-1, 2(2k-1), \dots, k(2k-1)\}$ if $n = 2k$, and $\{k, 3k, 5k, \dots, (2k+1)k\}$ if $n = 2k+1$.

Proof. By changing all vertex labels to their complements if necessary, we may assume that K_n has no more 0-vertices than 1-vertices. Thus let K_n have j vertices labeled 0, where $0 \leq j \leq n/2$. Then the other $(n-j)$ vertices of K_n are labeled 1. We have $e(0) = j(j-1)/2$ and $e(1) = (n-j)(n-j-1)/2$. Thus $|e(0) - e(1)| = \frac{1}{2}(n-1)(n-2j)$. Now let j exhaust all possible values. \square

Note. Theorem 3.9 shows that the maximum in Theorem 1.1 is attainable.

Corollary 3.10. For any $k \geq 1$, $\{0, 2k-1, 2(2k-1), \dots, k(2k-1)\}$ and $\{k, 3k, 5k, \dots, (2k+1)k\}$ are BI-representable.

Example 10. $BI(K_5 \cup N_5) = \{2, 6, 10\}$ and $BI(K_6 \cup N_6) = \{0, 5, 10, 15\}$.

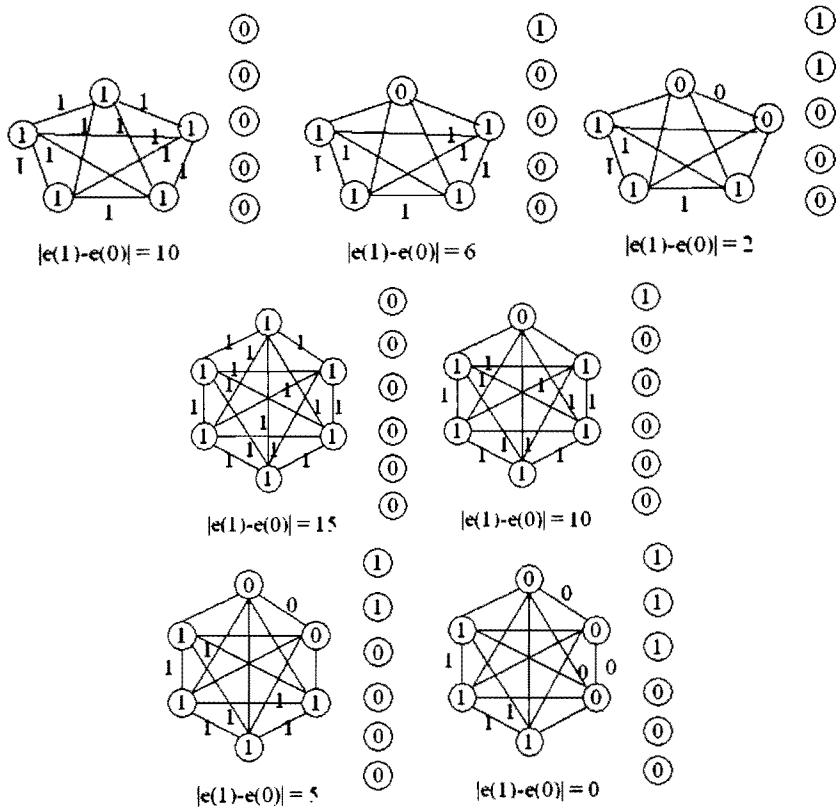


Figure 11.

4. Windmill graphs

Shee and Ho [7] considered one-point union of graphs which are cordial.

Notation. Let $K(m_1, m_2, \dots, m_n)$ be the one-point amalgamation of the complete graphs m_1, m_2, \dots, m_n vertices. Call the point at which the complete graphs are amalgamated the center of $K(m_1, \dots, m_n)$. If k of the m values are equal to the same value a , and if no confusion could arise, we use a^k to denote these values.

Theorem 4.1. For any $m \geq 0, n \geq 2$, $BI(K(n+2m+1, n)) = \{(2m+1)j - mn - 2m^2 \mid m \leq j \leq n+m-1\}$.

Proof. In $K(n+2m+1, n)$, there are $(2n+2m)$ vertices. Thus a friendly vertex labeling must have $v(0) = v(1) = n+m$. Without loss of generality, let the center be labeled 0. If K_{n+2m+1} has j non-center vertices labeled 0, then the other $(n+2m-j)$ vertices must be labeled 1. By friendliness, K_n has $(n+m-j-1)$ non-center vertices labeled 0, and the other $(j-m)$ vertices labeled 1. For these numbers to make sense, we must have $m \leq j \leq n+m-1$. Then in K_{n+2m+1} , $e(0) = (j+1)j/2$ and $e(1) = (n+2m-j)(n+2m-j-1)/2$, and in K_n , $e(0) = (n+m-$

$j)(n+m-j-1)/2$ and $e(1) = (j-m)(j-m-1)/2$. Algebraic calculations show that for the whole graph $K(n+2m+1, n)$, $e(0) - e(1) = (2m+1)j - mn - 2m^2$. \square

Example 11. $BI(K(5, 2)) = \{1, 2\}$.
 $K(5, 2)$

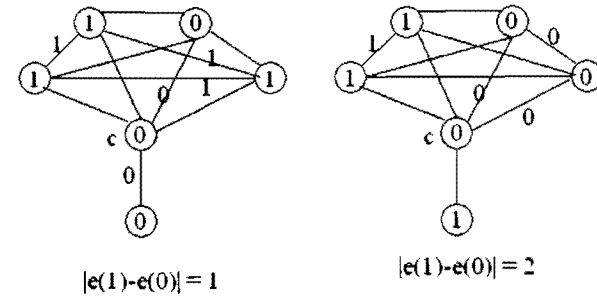


Figure 12.

Example 12. $BI(K(6, 3)) = \{1, 2, 4\}$.

$K(6, 3)$

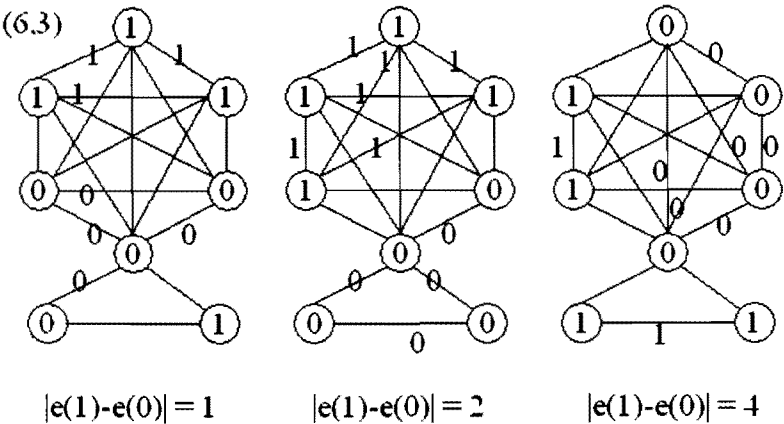


Figure 13.

Corollary 4.2. $BI(K(n+1, n)) = \{0, 1, \dots, n-1\}$.

Corollary 4.3. For any $n \geq 1$, the set $\{0, 1, \dots, n-1\}$ is BI-representable.

Theorem 4.4. For any $n \geq 3$, the balanced index set of the graph $K(n, n)$ is $\{0, n-1\}$.

Proof. Call the vertex at which the two copies of K_n are amalgamated the center of $K(n, n)$. Without loss of generality, let the center be labeled 0. There remain $(2n-2)$ vertices to be labeled.

Case 1: In one copy of K_n , m non-center vertices are labeled 0, while the other $(n-m-1)$ vertices are labeled 1, where $0 \leq m \leq (n-1)$. In the other copy of

K_n , $(n - m - 1)$ non-center vertices are labeled 0, while the other m vertices are labeled 1. In the first K_n , there are $(m + 1)$ vertices labeled 0, and $(n - m - 1)$ vertices labeled 1. Thus $e(0) = (m + 1)m/2$, and $e(1) = (n - m - 1)(n - m - 2)/2$. In the second K_n , there are $(n - m)$ vertices labeled 0, and m vertices labeled 1. Thus $e(0) = (n - m)(n - m - 1)/2$, and $e(1) = m(m - 1)/2$. Then for the entire $K_n(2)$, $e(0) = (m + 1)m/2 + (n - m)(n - m - 1)/2$, and $e(1) = (n - m - 1)(n - m - 2)/2 + m(m - 1)/2$, making $e(0) - e(1) = m + (n - m - 1) = n - 1$.

Case 2: In one copy of K_n , m non-center vertices are labeled 0, while the other $(n - m - 1)$ vertices are labeled 1, where $0 \leq m \leq (n - 2)$. In the other copy of K_n , $(n - m - 2)$ non-center vertices are labeled 0, while the other $(m + 1)$ vertices are labeled 1. In the first K_n , there are $(m + 1)$ vertices labeled 0, and $(n - m - 1)$ vertices labeled 1. Thus $e(0) = (m + 1)m/2$, and $e(1) = (n - m - 1)(n - m - 2)/2$. In the second K_n , there are $(n - m - 1)$ vertices labeled 0, and $(m + 1)$ vertices labeled 1. Thus $e(0) = (n - m - 1)(n - m - 2)/2$, and $e(1) = (m + 1)m/2$. Then for the entire $K(n, n)$, $e(0) = (m + 1)m/2 + (n - m - 1)(n - m - 2)/2$, and $e(1) = (n - m - 1)(n - m - 2)/2 + (m + 1)m/2$, making $e(0) - e(1) = 0$. \square

Corollary 4.5. For any $n \geq 3$, the set $\{0, n - 1\}$ is BI-representable.

Example 13. Figure 14 shows $BI(K(6,6)) = \{0, 5\}$.

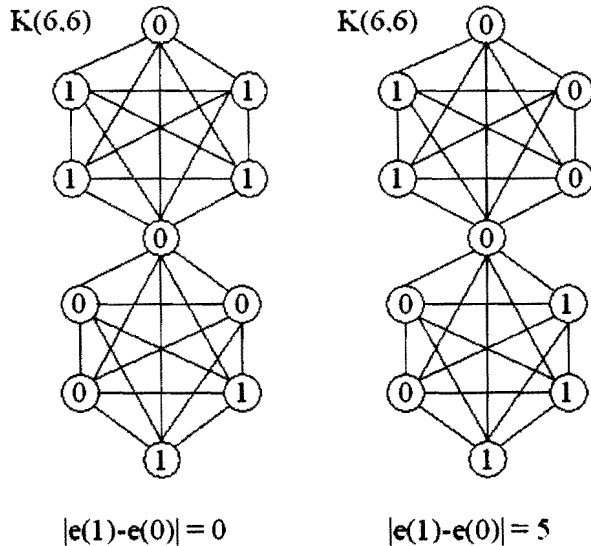


Figure 14.

5. Uniformly balanced graphs.

In [1], Chartrand, Zhang and the second author characterized graphs whose friendly index sets are subset of $\{0, 1\}$.

Recall that a graph G is balanced if there exists a binary labeling f such that $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$, or equivalently, $\min BI(G) \leq 1$.

Definition 4. A graph G is said to be *uniformly balanced* if it is balanced for any friendly binary labeling f .

Remark. A graph G is uniformly balanced if and only if $BI(G) \subseteq \{0, 1\}$.

In Section 3, we showed that the balance index set of the complete bipartite graph $BI(K_{m,m})$ is $\{0\}$. Thus the complete bipartite graph $K_{m,m}$ is uniformly balanced. We now show that the cycle C_n is uniformly balanced for all $n \geq 3$.

Lemma 5.1. For any (not necessarily friendly) vertex labeling of C_n , the differences $e(0) - e(1)$ and $v(0) - v(1)$ are the same.

Proof. Obviously it is impossible for all vertices to be labeled 0 or for all vertices to be labeled 1. In other words, there must be two adjacent vertices with complementary labels. Call them $v_{1,1}$ and $v_{1,0}$, with labels 1 and 0 respectively. Start from $v_{1,0}$, and go in the direction opposite to $v_{1,1}$. Consider the value of $c = (v(0) - v(1)) - (e(0) - e(1))$ as we traverse the cycle. When we start from $v_{1,0}$, $c = 1$. Let the next vertex labeled 1 be $v_{2,1}$. Before we reach $v_{2,1}$, $v(0)$ and $e(0)$ increase by the same amount, while $v(1)$ and $e(1)$ are unchanged. The edge leading to $v_{2,1}$ has no label. Thus at $v_{2,1}$, the value of c becomes 0. Continue to traverse the cycle in the same direction. Let the next vertex labeled 0 be $v_{2,0}$. Before we reach $v_{2,0}$, $v(1)$ and $e(1)$ increase by the same amount, while $v(0)$ and $e(0)$ are unchanged. The edge leading to $v_{2,0}$ has no label. Thus at $v_{2,0}$, the value of c becomes 1 again. Let the next vertex labeled 1 be $v_{3,1}$. The same argument shows that the value of c at $v_{3,1}$ is 0. Let the next vertex labeled 0 be $v_{3,0}$. The same argument shows that the value of c at $v_{3,0}$ is 1. Continue in this fashion. Eventually we will return to $v_{1,1}$ and $v_{1,0}$. At $v_{1,1}$, the value of c is 0. The edge leading from $v_{1,1}$ to $v_{1,0}$ has no label. Since the vertex label of $v_{1,0}$ has been counted when we start the process, we conclude that the value of $c = 0$ when we finish traversing the cycle. \square

Theorem 5.2. $BI(C_n) = \{0\}$ if n is even, and $\{1\}$ if n is odd.

Proof. As the vertex labeling is friendly, we have $|v(0) - v(1)| = 0$ when n is even, and $= 1$ when n is odd. The fact from Lemma 5.1 that $e(0) - e(1) = v(0) - v(1)$ finishes the proof. \square

Theorem 5.3. Let G be a 2-regular graph, i.e., G is the disjoint union of cycles. Then $BI(G) = \{0\}$ or $\{1\}$, if the number of vertices is even or odd respectively.

Proof. Assume that G is the disjoint union of k cycles. Let $c_i = (v(0) - v(1)) - (e(0) - e(1))$ for the i th cycle, where $i = 1, 2, \dots, k$, and v and e are the counts corresponding to that cycle. By Lemma 5.1, $c_i = 0$ for each i . Summing c_1, c_2, \dots, c_k , we have $(v(0) - v(1)) - (e(0) - e(1)) = 0$ for the whole graph G . Using the fact that the vertex labeling is friendly, we establish the result. \square

In [1], uniformly cordial graphs are completely characterized. However, at present the following problem is still unsolved.

Problem. Characterize uniformly balanced graphs.

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