All Trees of odd order with three even vertices are super edge-graceful

Sin-Min Lee
Department of Computer Science
San Jose State University
San Jose, California 95192
U.S.A.

Yong-Song Ho
Nan Chiao High School,
Singapore

ABSTRACT

A (p,q)-graph G is said to be **edge graceful** if the edges can be labeled by 1,2,…, q so that the vertex sums are distinct, mod p. It is shown that if a tree T is edge-graceful then its order must be odd. Lee conjectured that all trees of odd orders are edge-graceful. In [7], we establish that every tree of odd order with one even vertex is edge-graceful. Mitchem and Simoson [19] introduced the concept of super edge-graceful graphs which is a stronger concept than edge-graceful for trees. We show that every tree of odd order with three even vertices is super edge-graceful.

1.Introduction. All graphs in this paper are simple graphs. A graph with p vertices and q edges is **graceful** if there is an injective mapping f from the vertex set V(G) into {0,1,2,…,q} such that the induced map f*: E(G)→{1,2,…,q} defined by f*(e)= |f(u)-f(v)| where e=(u,v), is surjective.

Graceful graph labelings were first introduced by Alex Rosa [20] (around 1967) as means of attacking the problem of cyclically decomposing the complete graph into other graphs. Since Rosa's original article, literally hundreds of papers have been written on graph labelings (see [3]). A well-known conjecture due to Ringel and Kotzig is that all trees are graceful. This notorious conjecture is still unsolved. Rosa [20] showed that all caterpillars are graceful and also that all trees on at most 16 vertices are graceful.

Another dual concept of graceful labeling on graphs, edge-graceful labeling, was introduced by S.P. Lo [17] in 1985. G is said to be **edge-graceful** if the edges are labeled by 1,2,3,…,q so that the vertex sums are distinct, mod p.

Figure 1 shows a grid with 12 vertices and 17 edges with two different edge-graceful labelings.

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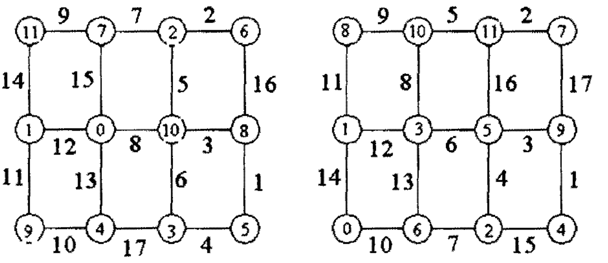


Figure 1.

A necessary condition of edge-gracefulness is (Lo [17])

$$q(q+1)\equiv \frac{p(p-1)}{2} \pmod p \qquad (1)$$

Apply (1) we have

Theorem 1. (Lee, Lee, Murty [6]) If G is a (p,q)-graph with $p\equiv 2 \pmod 4$ then G is not edge-graceful.

The following tantalizing conjecture is proposed in [8]

Conjecture 1: The Lo condition (2) is sufficient for a connected graph to be edge-graceful.

A sub-conjecture of the above (Lee [7]) has also not yet been proved:

Conjecture 2: All odd-order trees are edge-graceful.

In [1,3,4, 6,25] several classes of trees of odd orders are proved to be edge-graceful.

J. Mitchem and A. Simoson [19] introduced the concept of super edge-graceful graphs which is a stronger concept than edge-graceful for some classes of graphs. A graph H=(V,E) of order p and size q is said to be **super edge-graceful** if there exists a bijection

$f: E\rightarrow\{0, +1,-1,+2,-2,\ldots,(q-1)/2, -(q-1)/2\}$ if q is odd

$f: E\rightarrow\{ +1,-1,+2,-2,\ldots,(q-1)/2, -(q-1)/2\}$ if q is even

such that the induced vertex labeling f^* defined by $f^*(u)= \Sigma f(u,v)$: $(u,v) \varepsilon E$ }has the property:

$f^*: V\rightarrow\{0,+1,-1,\ldots,+(p-1)/2,-(p-1)/2\}$ if p is odd

$f^*: V\rightarrow\{+1,-1,\ldots,+p/2,-p/2\}$ if p is even

is a bijection.

Example 1. The following figure shows that the tree of order 7 is super edge-graceful with two different labelings.

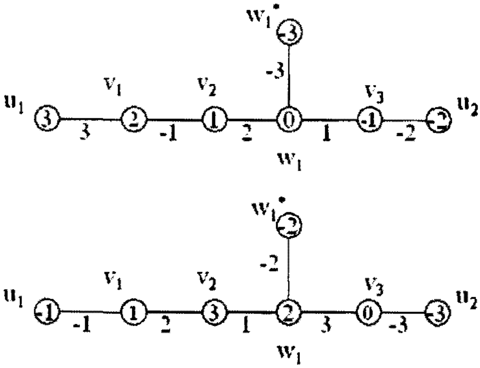


Figure 2

Not all trees have a super edge-graceful labeling.

Example 2. The following five trees of order 8 are not super edge-graceful and the other six are super edge-graceful.

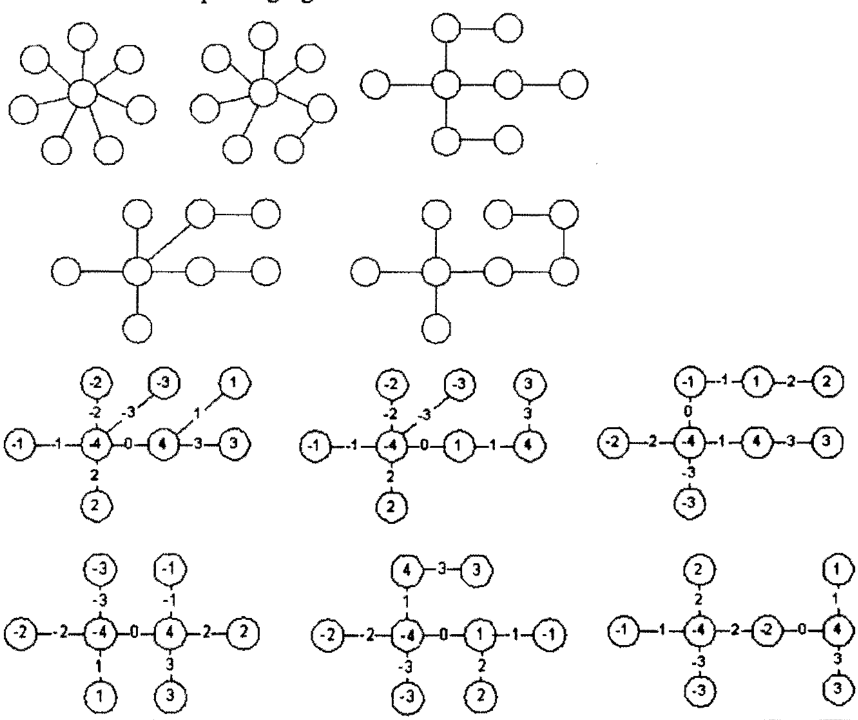


Figure 3.

We can view the super edge-graceful property as follows: For the graph G with vertex set V(G) and edge set E(G) with p=|V(G)| and q=|E(G)|, let (l, l^*) be *a function pair* which assigns integer labels to the edges and vertices; that is, l: E(G) →Z, and l^*: V(G) → Z. Following Lo [6], define G as edge-graceful if there is a function pair (l, l^*) such that l is onto {1, ...,q) and l^* is onto {0,..., p-1}, and

$$l^*(v) = \Big(\sum_{uv \in E(G)} l(uv) \Big) \bmod p.$$

Let

$$Q = \begin{cases} \{\pm 1, \ldots, \pm q/2\}, & \text{if q is even,} \\ \{0, \pm 1, \ldots, \pm (q-1/2)\}, & \text{if q is odd,} \end{cases}$$

$$P = \begin{cases} \{\pm 1, \ldots, \pm p/2\}, & \text{if p is even,} \\ \{0, \pm 1, \ldots, \pm (p-1/2)\}, & \text{if p is odd,} \end{cases}$$

Dropping the modularity operator and pivoting on symmetry about zero, define a graph G as a ***super edge-graceful graph*** if there is a function pair (l, l^*) such that l is onto Q and l^* is onto P, and

$$l^*(v) = \sum_{uv \in E(G)} l(uv)$$

Mitchem and Simoson [19] showed that

Theorem 2. If G is a super-edge-graceful graph and

$$q \equiv \begin{cases} -1 \pmod p, & \text{if q is even} \\ 0 \pmod p, & \text{if q is odd} \end{cases}$$

then G is also edge-graceful.

From which we conclude

Corollary 3. If G is super edge-graceful tree of odd order then it is edge-graceful.

If a tree G is super edge-graceful with labelings pair (l, l^*). We define f: E(G) → {1,2,...,p-1} by setting f(e) = l(e) if l(e) is positive and f(e) = p +l(e) if l(e) is negative. Then the edge-labeling f is edge-graceful. Figure 3 illustrates how to convert a super edge-graceful tree of order 7 into an edge-graceful tree.

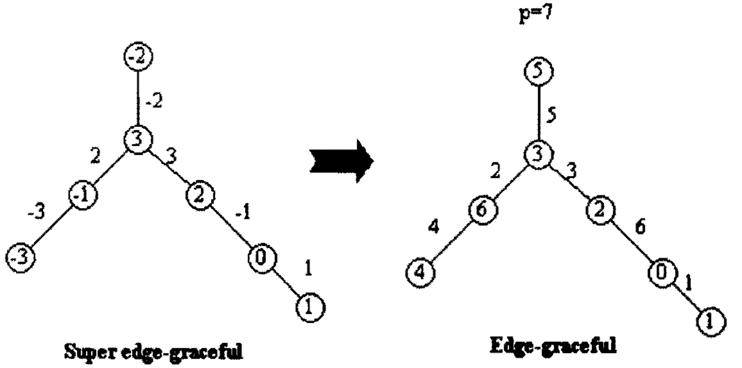


Figure 4.

In [19] Mitchem and Simoson showed that

A Growing Tree Algorithm. Let T be a super-edge-graceful tree with 2n edges. If any two vertices are added to T such that both are adjacent to a common vertex of T, then the new tree T* is also super-edge-graceful.

We reverse the above process and define a type of reducibility. For a tree T we delete all sets of even numbers of leaves incident with the same vertex and generate a new tree T*. Continue with the deletion process until no such sets of even number of leaves can be found. The final tree is said to be an **irreducible part** of T and denote it by irr(T). Now we define

Definition 1. A tree T is said to be **irreducible** if T = irr(T).

Example 3. The Figure 5 shows the irreducible part irr(T) of a tree of 14 vertices.

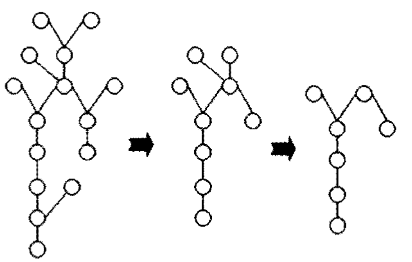


Figure 5.

For the class of trees of odd orders, to check a tree T is edge-graceful can be simplified by consider super edge-gracefulness of its irreducible part T.

Definition 2. Given a tree T, a vetex u is said to be **even** if its degree is even.

In [7], every odd tree with a single even vertex is shown to be edge-graceful. However, a tree with two even vertices may or may not super edge-graceful. The path P_4 has two even vertices is not super vertex-graceful. The following example shows that there exists a super edge-graceful tree with two even vertices.

Example 4. The tree Y(3,5) with 10 vertices and two even vertices is super edge-graceful.

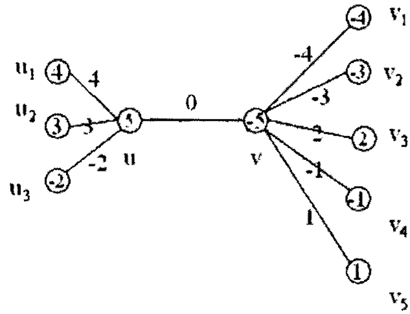


Figure 6.

Finding the edge-graceful labelings or super edge-graceful labelings of graphs are related to solving systems of linear Diophantine equations. Several classes of graphs had been shown to be edge-graceful. The interested reader can consult ([9,10,11, 12, 13,14,15,16,17,18, 19,20, 21, 22,23,24]).

2.Irreducible trees of odd orders with three even vertices not on the same path.

A tree is called a **spider** if it has a center vertex c of degree k >1 and all the other vertex is either a leaf or with degree 2. Thus a spider is an amalgamation of k paths with various lengths. If it has x1's path of length a1, x2's path of length x2,..., we shall denote the spider by $SP(a_1^{x1}a_2^{x2},,,a_m^{xm})$ where $a_1 < a_2 < \ldots < a_m$ and x1+x2+..+xm=k. (see Figure 7).

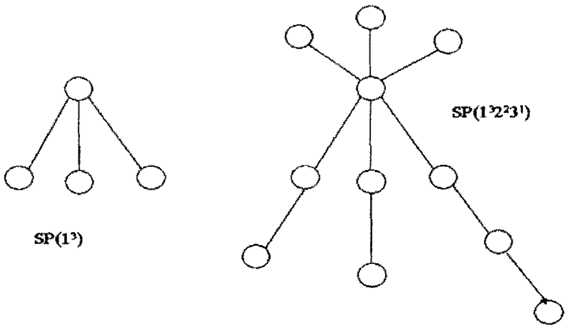


Figure 7

Keene and Simoson [4] showed that

Theorem 2.1. All three legged spiders of odd order are edge-graceful.

Theorem 2.2. Any four-legged spider of odd order, with two legs of equal length, is edge-graceful.

Mitchem and Simoson [19] also showed that

Theorem 2.3 Every regular spider of odd order is super-edge –graceful .

In [16], we show that an irreducible tree T of odd order and of diameter 4 is of the form Sp(2^k) for some integer k≥ 2. They are super edge-graceful. From that we can show that all trees of odd order of diameter at most four are edge-graceful.

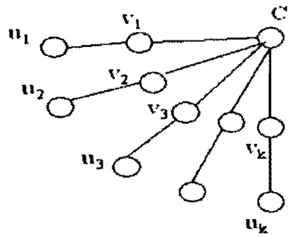


Figure 8.

Theorem 2.4. An irreducible tree of odd order with three even vertices that are not on the same path is of the form SP(2,2,2). It is super edge-graceful as shown as Figure 9.

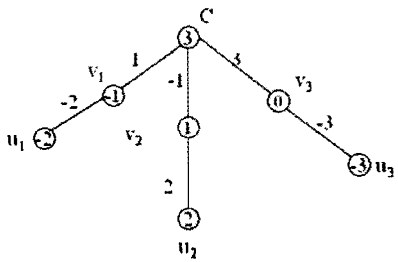


Figure 9.

3.Irreducible trees of odd orders with three even vertices on the same path.

In this section, we consider irreducible trees of odd orders with three even vertices on the same path. There are three possible cases:

Case 1. Three even vertices are adjacent together.

Theorem 3. An irreducible tree of odd order with three even vertices.that are on the same path and are adjacent together is of the form P_5. It is super edge-graceful.

Figure 10 shows a super edge-graceful labeling of P_5.

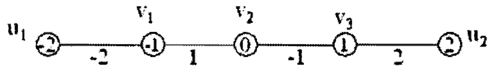


Figure 10.

Case 2. Two even vertices are adjacent together.

Theorem 4. An irreducible tree of odd order with three even vertices.that are on the same path and two of them are adjacent together is of the form F(k) with $k \geq 1$.

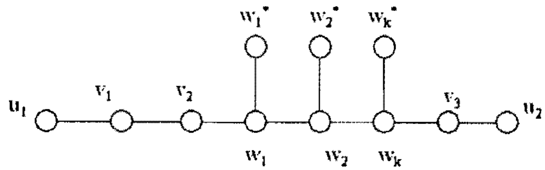


Figure 11.

Theorem 5. An irreducible tree of odd order with three even vertices.that are on the same path and none of them are adjacent together is of the form F(t,k) with $t \geq 1$ and $k \geq 1$.

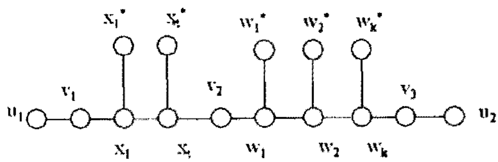


Figure 12.

4.Super Edge-gracefulness of F(k).

Theorem 6. The tree F(k) is super edge-graceful for all $k \geq 1$.

Proof. The tree F(k) has order p = 5 +2k and size q = 4 +2k. We label the edges along the (u_i,u_2)-path by {-(2+k), 2+k,-(1+k), k,...,-2, (2+k)}.

All the pendant edges of $w_1,w_2,..,w_{k-1}$, w_k are labeled by {k, k-1,...,2,-1} respectively.

We see that it is a super edge-graceful labeling.

Example 5. Figure 13 gives super edge-graceful labelings of F(1), F(2) and F(3).

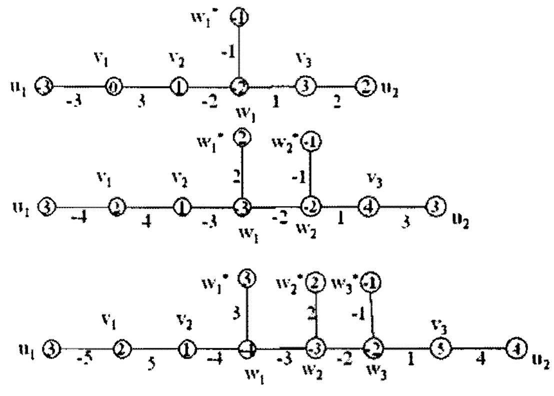


Figure 13.

5.Super Edge-gracefulness of F(t,k).

Theorem 7. The tree F(t, k) is super edge-graceful for all $t, k \geq 1$.

Proof. The tree F(t, k) has order $p = 5 + 2(k + t)$ and size $q = 4 + 2(k+t)$. We label the edges along the (u_i, u_2)-path by $-(2+t+k)$, $2+t+k$ for edges (u_1,v_1), (v_1,x_1), and (x_1,x_2), (x_2,x_3),..,(x_{k-1},x_k) (x_k, v_3), by $-(1+t+k)$, $-(t+k)$, $-(t+k-1)$,…, $-(k+3)$, $-(k+2)$. Then we label (v_3,w_1), (w_1,w_2), …, (w_k,v_3), (v_3,u_2) by $k+1$, $-k$, $-(k-1)$,…, $-3,-2,-1$, $-(k+1)$.

All the pendant edges of $x_1,x_2,..,x_{t-1}$, x_t are labeled by $\{(t+k)+1, t+k, \dots k+3, k+2\}$ and the pendant edges of $w_1,w_2,..,w_{k-1}$, w_k are labeled by $\{k, k-1, k-2, \dots, 2, 1\}$ respectively.

We see that it is a super edge-graceful labeling.

Example 6. Figure 14 gives super edge-graceful labelings of F(1,1) and F(2,3)

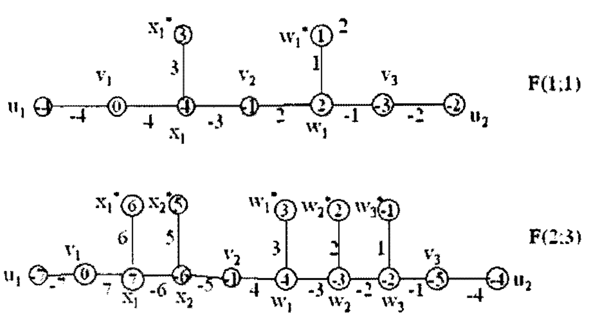


Figure 14.

Thus we obtain

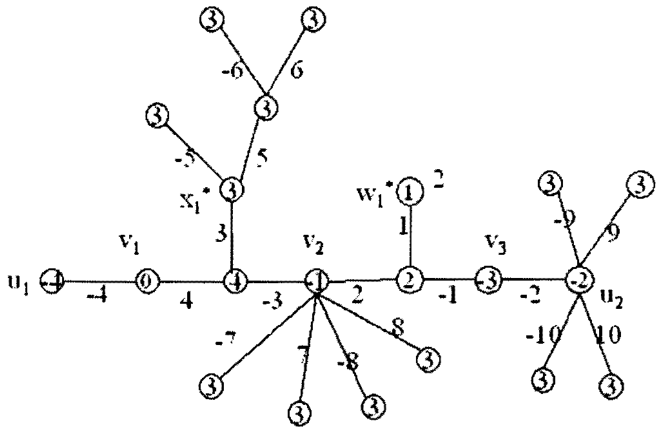
Theorem 8 . All trees of odd order with three even vertices are edge-graceful.

In particular, we have

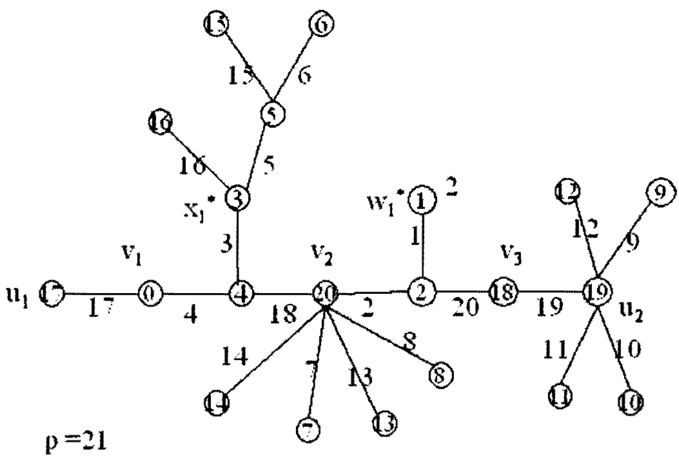
Corollary 9. All trees of odd order with three even vertices are edge-graceful.

Using the result we obtain in this paper, we can find the edge-graceful labeling of odd trees with three even vertices easily. We illustrate the method with an example of tree of order 21.

Example 7. T is of order 21, we see its irreducible part is F(1,1). First of all we find a super edge-graceful labeling of F(1,1) and then extend it to T. Now we can convert the super edge-graceful labeling directly to an edge-graceful labeling of T. (Figure 15).



T is super edge-graceful



T is edge-graceful

Figure 15.

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