

# On super edge-magic graphs which are weak magic

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## ABSTRACT

A  $(p,q)$  graph  $G$  is *total edge-magic* if there exists a bijection  $f: V \cup E \rightarrow \{1, 2, \dots, p+q\}$  such that for each  $e=(u,v)$  in  $E$ , we have  $f(u) + f(e) + f(v)$  as a constant. For a graph  $G$ , denote  $M(G)$  the set of all total edge-magic labelings. The magic strength of  $G$  is the minimum of all constants among all labelings in  $M(G)$ , and denoted by  $emt(G)$ . The maximum of all constants among  $M(G)$  is called the maximum magic strength of  $G$  and denoted by  $eMt(G)$ . Hegde and Shetty classify a magic graph as **strong** if  $emt(G) = eMt(G)$ , **ideal magic** if  $1 \leq eMt(G) - emt(G) \leq p$  and **weak magic**, if  $eMt(G) - emt(G) > p$ . A total edge-magic graph is called a *super edge-magic* if  $f(V(G)) = \{1, 2, \dots, p\}$ . The problem of identifying which kinds of super edge-magic graphs are weak-magic graphs is addressed in this paper.

**1. Introduction.** A  $(p,q)$ -graph  $G = (V, E)$  with  $p$  vertices and  $q$  edges is called *total edge magic* if there is a bijection  $f: V \cup E \rightarrow \{1, 2, \dots, p+q\}$  such that there exists a constant  $s$  for any  $(u,v)$  in  $E$   $f(u) + f(u,v) + f(v) = s$ . The original concept of a total edge-magic graph is credited to Kotzig and Rosa [12,13]. Originally, they termed it a magic graph. Motivated by the definition of total edge-magic labelings, Enomoto, Llado, Nakamigawa and Ringel [4] introduced the concept of super edge-magic graphs in 1998. A total edge-magic graph  $G$  is called *super edge-magic* if  $f(V(G)) = \{1, 2, \dots, p\}$ .

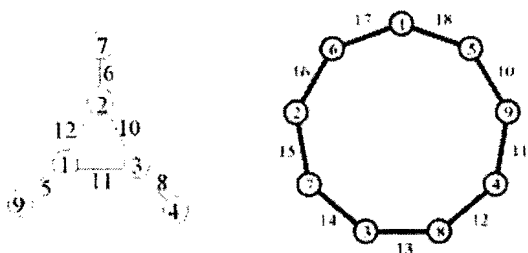


Figure 1.

The figure 1 shows a unicyclic graphs with 6 vertices which permits a total edge-magic labeling and the cycle  $C_9$  with a super edge-magic labeling. In fact, in [12] Kotzig and Rosa showed that every caterpillar and  $(2k+1) K_2$  are super edge-magic. However, the edge-magic labeling of even cycles [12] is not super edge-magic. In [4], Enomoto et. al. gave a super edge-magic labeling for odd cycles. Independently, Craft and Tesar [3], Godbold and Slater [10] rediscovered that all cycles are total edge-magic.

A subset  $S$  of integers is called *consecutive* if  $S$  consists of consecutive integers. Chen [2] showed that a graph  $G$  is super edge-magic if and only if there exists a vertex labeling  $f$  such that the two sets  $f(V(G))$  and  $\{f(u)+f(v) : (u,v) \in E(G)\}$  are both consecutive. Independently, Figueroa-Centeno et. al. [6,8] have also obtained the same result. They have shown that if  $f:V(G) \rightarrow \{1,2,\dots,p\}$  is a bijection of a  $(p,q)$ -graph  $G$  and  $S = \{f(u)+f(v) : (u,v) \in E\}$  is consecutive with  $s = \min(S)$ , then  $f$  can be extended to a super edge-magic labeling of  $G$  defined by  $f((u,v)) = p+q+s-f(u)-f(v)$  for all edge  $(u,v)$  of  $E(G)$ . In light of this result, it suffices to exhibit the vertex labeling of a super edge-magic graph.

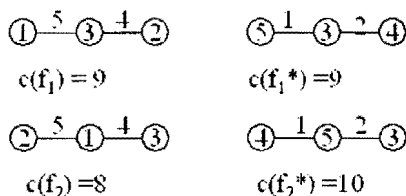
Numerous classes of graphs have been identified total edge-magic or super edge-magic. [2,4,7,9,10,14,15,16,17]

For a graph  $G$ , let  $M(G)$  denote the set of all total edge-magic labelings. If  $f$  is in  $M(G)$  with magic sum  $k$ , then its dual  $f^*$  which is defined by  $f^*(x) = (p+q+1) - f(x)$  for each  $x$  in  $V(G)$  and  $f^*((x,y)) = (p+q+1) - f((x,y))$  for each  $(x,y)$  in  $E(G)$  is also in  $M(G)$  with magic sum  $3(p+q+1) - k$ .

The **magic strength** of  $G$  is the minimum of all constants among all labelings in  $M(G)$ , and is denoted by  $emt(G)$ . The maximum of all constants among  $M(G)$  is called the **maximum magic strength** of  $G$  and is denoted by  $eMt(G)$ . Hegde and Shetty [11] classify a magic graph as **strong** if  $emt(G) = eMt(G)$ , **ideal** magic if  $1 \leq eMt(G) - emt(G) \leq p$  and **weak magic** if  $eMt(G) - emt(G) > p$ .

**Example 1.**  $P_2$  is strong magic, for we have  $M(P_2) = \{f_1, f_2\}$  where  $f_1(v_1) = 1$ ,  $f_1(v_2) = 2$  and  $f_2(v_1) = 2$ ,  $f_2(v_2) = 1$ .

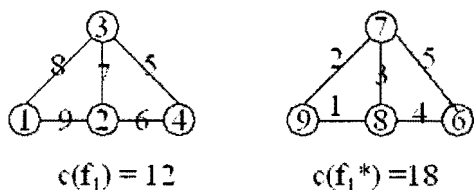
**Example 2.**  $P_3$  is ideal magic.



**Figure 2.**

**Example 3.** Godbold and Slater [10] have shown that  $eMt(C_{2n+1}) = 7n+5$ ,  $emt(C_{2n+1}) = 5n+4$ . As  $eMt(C_{2n+1}) - emt(C_{2n+1}) = 2n+1 = p(C_{2n+1})$ . Thus all odd cycles are ideal magic.

**Example 4.**  $P_3 + K_1$  is weak magic. To see that the graph is weak magic, it suffices to have a total edge-magic labeling  $f$ , such that its dual  $f^*$  has the property that  $c(f^*) - c(f) > p$ . Figure 3 illustrates such a labeling exists for  $P_3 + K_1$ .



**Figure 3.**

The problem of identifying which kinds of super edge-magic graphs are weak-magic graphs is addressed in this paper. We present here eight families, out of many, super edge-magic graphs which are weak magic.

## 2. Super edge-magic graphs which are weak magic.

**Family 1.** Graph  $\Xi(n)$ ,  $n \geq 1$

For  $n \geq 1$ , let  $\Xi(n)$  be the graph with  $V(\Xi(n)) = \{c_1, c_2, \dots, c_6\} \cup \{v_1, v_2, \dots, v_n\}$  and  $E(\Xi(n)) = \{(c_1, c_2), (c_2, c_3), (c_3, c_4), (c_4, c_5), (c_5, c_6), (c_1, c_6), (c_1, c_4)\} \cup \{(c_4, v_t) : t=1, 2, \dots, n\}$ . We see that  $p=n+6$  and  $q = n+7$ . (see Figure 4).

**Theorem 1.** The graph  $\Xi(n)$  is super edge-magic and weak magic for all  $n \geq 1$ .

**Proof.** Let us define  $f: V(\Xi(n)) \cup E(\Xi(n)) \rightarrow \{1, 2, 3, \dots, 2n+13\}$  as follows:

$$f(c_1) = 3, f(c_2) = 2, f(c_3) = 6, f(c_4) = 4, f(c_5) = 5, f(c_6) = 1,$$

$$f(v_t) = t+6, t = 1, \dots, n, \text{ and } f((c_4, v_t)) = 2n+7-t, t = 1, \dots, n.$$

$$f((c_1, c_2)) = 2n+12, f((c_2, c_3)) = 2n+9, f((c_3, c_4)) = 2n+7, f((c_4, c_5)) = 2n+8,$$

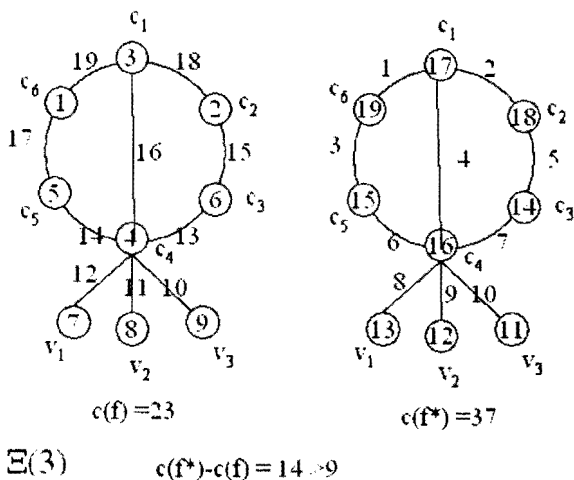
$$f((c_5, c_6)) = 2n+11, f((c_6, c_1)) = 2n+13, f((c_1, c_4)) = 2n+10.$$

It can be easily checked that  $f$  is a super edge-magic with magic sum  $c(f) = 2n+17$ .

Now let  $f^*$  be the dual of  $f$  and its magic sum is  $c(f^*) = 3(p+q+1) - (2n+17) = 3(2n+14) - (2n+17) = 4n+25$ .

We see that  $eMt(G) - emt(G) = c(f^*) - c(f) = 2n+8 > n+6 = p(\Xi(n))$ . Thus  $\Xi(n)$  is weak magic.

**Example 5.** Super edge-magic labelings of  $\Xi(3)$  in Figure 4.



**Figure 4.**

**Family 2.** Let  $P_{2n}(+)N_m$  be the graph with  $p=2n+m$  and  $q = 2(m+n) - 1$ .

$V(P_{2n}(+)N_m) = \{v_1, v_2, \dots, v_{2n}, y_1, y_2, \dots, y_m\}$  where  $V(P_{2n}) = \{v_1, v_2, \dots, v_{2n}\}$  and  $V(N_m) = \{y_1, y_2, \dots, y_m\}$  and  $E(P_{2n}(+)N_m) = E(P_{2n}) \cup \{(v_i, y_1), (v_i, y_2), \dots, (v_i, y_m), \dots, (v_{2n}, y_1), (v_{2n}, y_2), \dots, (v_{2n}, y_m)\}$  (see Figure 5).

**Theorem 2.**  $P_{2n}(+)N_m$  is super edge-magic and weak magic for all  $n, m > 1$ .

**Proof.** Let us define  $f: V(P_{2n}(+)N_m) \rightarrow \{1, 2, 3, \dots, 2n+m\}$  as follows:

$$f(v_{1+2t}) = 1 + t, \quad t = 0, 1, \dots, n-1$$

$$f(v_{2+2t}) = m + n + 1 + t, \quad t = 0, 1, \dots, n-1$$

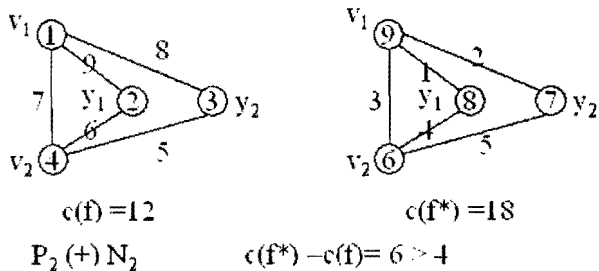
$$f(y_k) = k + n, \quad k = 1, 2, \dots, m$$

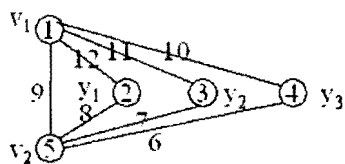
It can be easily checked that  $f(E(P_{2n}(+)N_m))$  is a consecutive set with the magic sum  $c(f) = 5n+3m+1$  and  $c(f^*) = 6m+7n-1$ .

We see that  $eMt(G) - emt(G) \geq c(f^*) - c(f) = 2n + 3m - 2 > 2n + m = p(P_{2n}(+)N_m)$ .

Thus  $P_{2n}(+)N_m$  is weak magic.

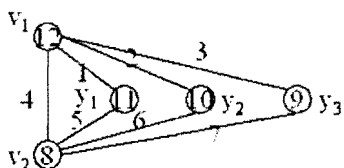
**Example 6.** Super edge-magic labeling of  $P_2(+)N_m$  for  $m = 2, 3$  and  $P_4(+)N_2$  in Figure 5.





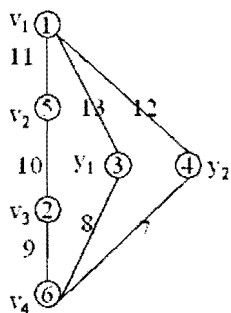
$$c(f) = 15$$

$P_2(+)N_3$



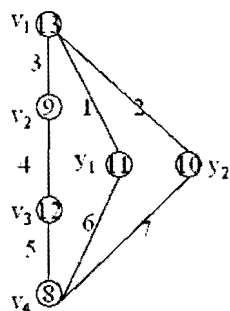
$$c(f^*) = 24$$

$$c(f^*) - c(f) = 9 > 5$$



$$c(f) = 17$$

$P_2(+)N_2$



$$c(f^*) = 25$$

$$c(f^*) - c(f) = 8 > 6$$

**Figure 5.**

**Family 3.** The planar graph  $(P_2 \cup k K_1) + N_2$

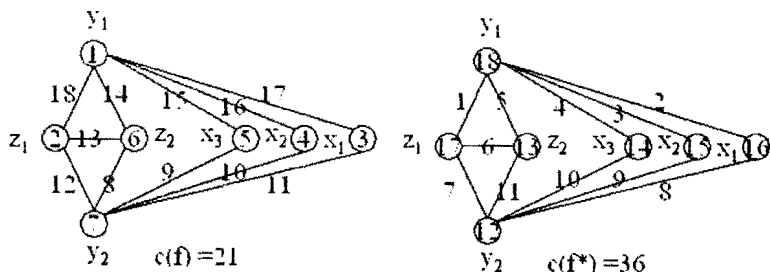
**Theorem 3.** For  $k > 1$ , the planar graph  $(P_2 \cup k K_1) + N_2$  is weak magic.

**Proof.** Let the vertex set of  $P_2 \cup k K_1$  be  $\{z_1, z_2, x_1, \dots, x_k\}$  and  $V(N_2) = \{y_1, y_2\}$ . We have  $p = k + 4$  and  $q = 2k + 5$ .

Define a labeling  $f: V((P_2 \cup k K_1) + N_2) \rightarrow \{1, 2, \dots, k + 4\}$  by  $f(y_1) = 1$ ,  $f(y_2) = k + 4$ ,  $f(z_1) = 2$ ,  $f(z_2) = k + 3$  and  $f(x_s) = s + 2$  for  $s = 1, 2, \dots, k$ .

It is clear that  $f$  induces a consecutive labeling on the edges. Therefore  $(P_2 \cup k K_1) + N_2$  is super edge-magic with magic constant  $c(f) = 3k + 12$ . As  $c(f^*) = 6k + 18$  and  $c(f^*) - c(f) = 3k + 6 > p((P_2 \cup k K_1) + N_2)$ . We conclude that  $(P_2 \cup k K_1) + N_2$  is weak magic.

**Example 6.** We give a super edge-magic labeling of  $(P_2 \cup 3 K_1) + N_2$  which shows that it is weak magic.



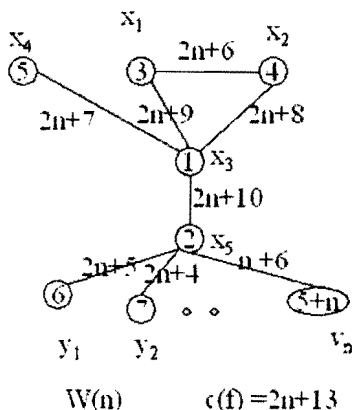
$$(P_2 \cup 3 K_1) + N_2$$

$$c(f^*) - c(f) = 15 > 7$$

Figure 6.

**Family 4. The unicyclic graph  $W(n)$**

For any integer  $n \geq 1$ , the unicyclic graph  $W(n)$  is the graph with vertex set  $V(W(n)) = \{x_1, x_2, x_3, x_4, x_5, y_1, y_2, \dots, y_n\}$  and the edge set  $E(W(n)) = \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_5), (x_5, x_1)\} \cup \{(x_i, y_i) : i = 1, 2, \dots, n\}$  (see Figure 7).



$$W(n) \quad c(f) = 2n + 13$$

Figure 7.

**Theorem 4.** For any integer  $n \geq 1$ , the graph  $W(n)$  is super edge-magic and weak edge-magic.

**Proof.** The graph  $W(n)$  has  $5 + n$  vertices and  $5+n$  edges.

Define a labeling  $f: V(W(n)) \rightarrow \{1, 2, \dots, 5 + n\}$  by

$$f(x_1) = 3, f(x_2) = 4, f(x_3) = 1, f(x_4) = 5, f(x_5) = 2 \text{ and } f(y_i) = 5 + i, \text{ for } i = 1, \dots, n.$$

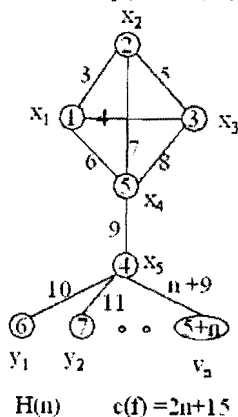
It is clear that  $f$  induces a consecutive labeling on the edges. Thus  $f$  is a super edge-magic labeling with

$$c(f) = 2n + 13.$$

Now  $c(f^*) - c(f) = 2n + 7 > n + 5 = p(W(n))$ . Therefore  $W(n)$  is weak magic.

**Family 5. The graph  $H(n)$ .**

For any integer  $n \geq 1$ , the graph  $H(n)$  is the graph with vertex set  $V(H(n)) = \{x_1, x_2, x_3, x_4, x_5, y_1, y_2, \dots, y_n\}$  and the edge set  $E(H(n)) = \{(x_1, x_2), (x_1, x_3), (x_1, x_4), (x_2, x_2), (x_2, x_4), (x_3, x_4), (x_4, x_5)\} \cup \{(x_5, y_i) : i = 1, 2, \dots, n\}$  (see Figure 8).



**Figure 8.**

**Theorem 5.** For any integer  $n \geq 1$ , the graph  $H(n)$  is super edge-magic and weak edge-magic.

**Proof.** The graph  $H(n)$  has  $5 + n$  vertices and  $7 + n$  edges.

Define a labeling  $f: V(H(n)) \rightarrow \{1, 2, \dots, 5 + n\}$  by

$f(x_1) = 1, f(x_2) = 2, f(x_3) = 3, f(x_4) = 5, f(x_5) = 4$  and  $f(y_i) = 5 + i$ , for  $i = 1, \dots, n$ .

It is clear that  $f$  induces a consecutive labeling on the edges. Thus  $f$  is a super edge-magic labeling with

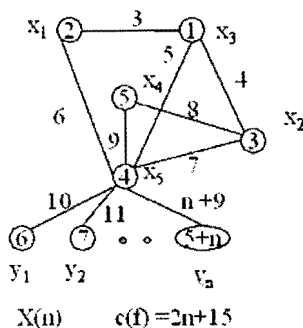
$$c(f) = 2n + 15.$$

Now  $c(f^*) - c(f) = 3(p+q+1) - 2c(f) = 2n + 9 > n + 5 = p(H(n))$ .

Therefore  $H(n)$  is weak magic.

**Family 6.** The graph  $X(n)$ .

For any integer  $n \geq 1$ , the graph  $X(n)$  is the graph with vertex set  $V(X(n)) = \{x_1, x_2, x_3, x_4, x_5, y_1, y_2, \dots, y_n\}$  and the edge set  $E(X(n)) = \{(x_1, x_2), (x_1, x_3), (x_2, x_3), (x_1, x_4), (x_2, x_4), (x_3, x_4), (x_4, x_5)\} \cup \{(x_5, y_i) : i = 1, 2, \dots, n\}$  (see Figure 9).



**Figure 9.**

**Theorem 6.** For any integer  $n \geq 1$ , the graph  $X(n)$  is super edge-magic and weak edge-magic.

**Proof.** The graph  $X(n)$  has  $5 + n$  vertices and  $12+n$  edges.

Define a labeling  $f: V(X(n)) \rightarrow \{1, 2, \dots, 5 + n\}$  by

$f(x_1) = 2, f(x_2) = 3, f(x_3) = 1, f(x_4) = 5, f(x_5) = 4$  and  $f(y_i) = 5 + i$ , for  $i = 1, \dots, n$ .

It is clear that  $f$  induces a consecutive labeling on the edges. Thus  $f$  is a super edge-magic labeling with

$$c(f) = 2n + 15.$$

Now  $c(f^*) - c(f) = 3(p+q+1) - 2c(f) = 2n + 9 > n + 5 = p(X(n))$ .

Therefore  $X(n)$  is weak magic.

**Family 7. The braid graph  $B(n)$ .**

We consider a family of graphs which we will call **braid graphs**. For each  $n > 2$ , the braid graph

$B(n)$  is defined as follows:

$$V(B(n)) = \{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n\} \text{ and}$$

$$E(B(n)) = \{(x_i, x_{i+1}) : i = 1, 2, \dots, n - 1\} \cup \{(y_i, y_{i+1}) : i = 1, 2, \dots, n - 1\} \cup \{(x_i, y_{i+1}) : i = 1, 2, \dots, n - 1\} \cup \{(y_i, x_{i+2}) : i = 1, 2, \dots, n - 2\}.$$

**Theorem 7.** The braid graph  $B(n)$  is weak magic for all  $n \geq 3$ .

**Proof.** The graph  $B(n)$  has  $2n$  vertices and  $4n - 5$  edges. We define a vertex labeling  $f$  on  $B(n)$  as follows:

$$f(x_i) = 2i - 1 \text{ for } i = 1, \dots, n$$

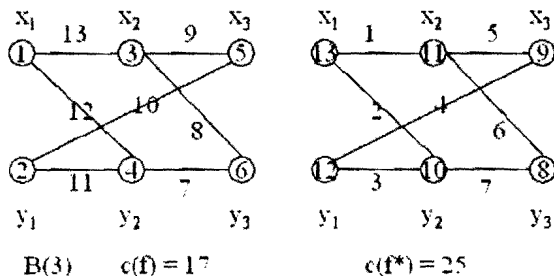
$$f(y_i) = 2i \text{ for } i = 1, \dots, n.$$

We see that  $f$  induces a consecutive labeling on the edge set (see Figure 10).

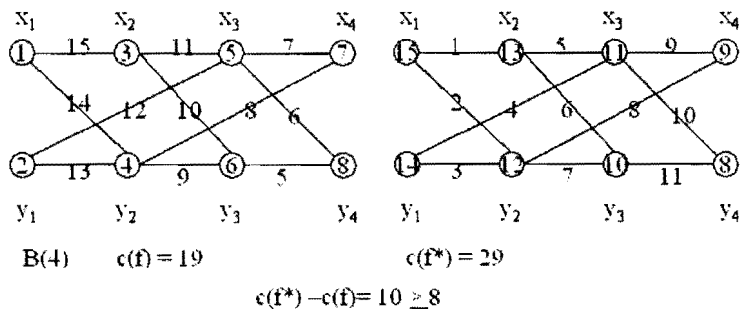
$$c(f) = 6n - 1 \text{ and } c(f^*) = 12n - 11$$

Therefore  $eMt(B(n)) - emt(B(n)) \geq c(f) - c(f^*) = 6n - 10 > 2n = p(B(n))$ . Thus  $B(n)$  is weak magic.

**Example 8.**



$$c(f^*) - c(f) = 8 > 6$$



**Figure 10.**

**Family 8.** Jellyfish graph  $J(m,n)$ .

For integers  $m, n > 0$ , we consider the graph  $J(m, n)$  with vertex set  $V(J(m, n)) = \{u, v, x, y\} \cup \{x_1, x_2, \dots, x_m\} \cup \{y_1, y_2, \dots, y_n\}$  and edge set  $E(J(m, n)) = \{(u, x), (u, v), (u, y), (v, x), (v, y)\} \cup \{(x_i, x) : i = 1, 2, \dots, m\} \cup \{(y_i, y) : i = 1, 2, \dots, n\}$ .

We will refer to  $J(m,n)$  as a Jellyfish graph. We see that  $p = m + n + 4$  and  $q = m + n + 5$ .

**Theorem 8.** The Jellyfish graph  $J(m, n)$  is weak magic for all  $m, n > 0$ .

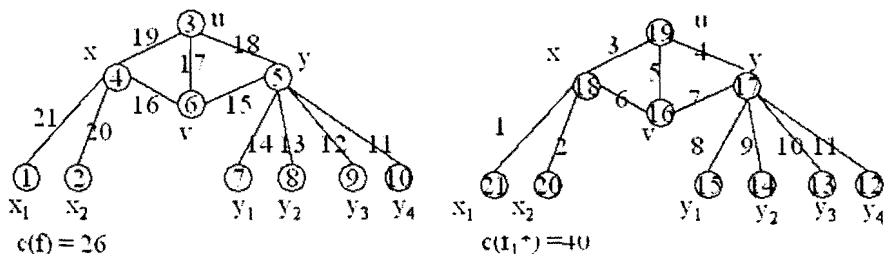
**Proof.** Without loss of generality we assume  $m \leq n$ . We define a labeling  $f: V(J(m, n)) \rightarrow \{1, 2, \dots, m+n+4\}$  as follows:

$$\begin{aligned}
 f(x_i) &= i \text{ for } i = 1, 2, \dots, m \\
 f(u) &= m + 1, f(x) = m + 2, f(y) = m + 3, f(v) = m + 4, \\
 f(y_i) &= m + 4 + i \text{ for } i = 1, \dots, n.
 \end{aligned}$$

We see that the labeling  $f$  induces a consecutive labeling on  $E(J(m, n))$ . Thus, we conclude that  $J(m, n)$  is super edge-magic.

As  $c(f) = 3m + 2n + 12$  and  $c(f^*) = 3m + 4n + 18$ . We have  $eMt(J(m, n)) - emt(J(m, n)) \geq c(f^*) - c(f) = 2n + 6 > m + n + 4 = p(J(m, n))$ . Thus  $J(m, n)$  is weak magic.

**Example 9.** A super edge-magic labeling of Jellyfish graph  $J(2, 4)$  is shown as follows:



$J(2,4) \quad c(f^*) - c(f) = 14 > 10$

**Figure 11.**

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