ON THE PRODUCTS OF GROUP–MAGIC GRAPHS

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This paper is dedicated to the memory of Gwong C. Sun.

Abstract. Let A be an abelian group. We call a graph $G = (V, E)$ A–**magic** if there exists a labeling $f : E(G) \to A - \{0\}$ such that the induced vertex set labeling $f^+ : V(G) \to A$, defined by $f^+(v) = \Sigma f(u, v)$ where the sum is over all $(u, v) \in E(G)$, is a constant map. For four classical products, we examine the A–magic property of the resulting graph obtained from the product of two A–magic graphs.

1. Introduction

Let G be a connected graph without multiple edges or loops. For any abelian group A (written additively), let $A^* = A - \{0\}$. A function $f : E(G) \to A^*$ is called a *labeling* of G. Any such labeling induces a map $f^+ : V(G) \to A$, defined by $f^+(v) = \Sigma f(u, v)$ where the sum is over all $(u, v) \in E(G)$. If there exists a labeling f whose induced map on $V(G)$ is a constant map, we say that f is an *A–magic labeling* and that G is an *A–magic* graph.

Z–magic graphs were considered by Stanley [16,17], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous diophantine equations. Doob [1,2,3] and others [6,9,11] have studied A–magic graphs and Z_k–magic graphs were investigated in [4,7,8,10]. The construction of magic graphs was studied by Sun and Lee [18]. In this paper, we extend some results to A–magic graphs. In particular, graph products offer a straight–forward and systematic means of constructing A–magic graphs.

Within the mathematical literature, various definitions of magic graphs have been introduced. The original concept of an A–magic graph is due to J. Sedlacek [14,15], who defined it to be a graph with real–valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices. Previously, Kotzig and Rosa [5] had introduced yet another definition of a magic graph. Over the years, there has been great research interest in graph labeling problems. The interested reader is directed to Wallis' [19] recent monograph on magic graphs.

2. Definitions and examples

For any commutative ring R with unity, $U(R)$ denotes the multiplicative group of units in R. The product of two graphs $G_1(p_1, q_1) = (V_1, E_1)$ and $G_2(p_2, q_2) =$

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(V_2, E_2) can be defined in various ways. Within the product, the vertices will be denoted by $(a, b) : a \in V_1$ and $b \in V_2$, and the edges will be denoted by $((a, b), (a', b')) : a, a' \in V_1$ and $b, b' \in V_2$. Let us recall the following definitions of various products of graphs.

Definition 1. Cartesian product $G_1 \times G_2$: $V(G_1 \times G_2) = V_1 \times V_2 = \{(a, b) : a \in V_1 \ \wedge \ b \in V_2\}$ and $E(G_1 \times G_2) = \{((a, b), (a', b')) : a, a' \in V_1 \ \wedge \ b, b' \in V_2 \ \wedge \ ((a = a' \ \wedge \ (b, b') \in E_2) \ \vee \ (b = b' \ \wedge \ (a, a') \in E_1))\}$.

Definition 2. Lexicographic product $G_1 \bullet G_2$: $V(G_1 \bullet G_2) = V_1 \times V_2 = \{(a, b) : a \in V_1 \ \wedge \ b \in V_2\}$ and $E(G_1 \bullet G_2) = \{((a, b), (a', b')) : a, a' \in V_1 \ \wedge \ b, b' \in V_2 \ \wedge \ ((a = a' \ \wedge \ (b, b') \in E_2) \ \vee \ (a, a') \in E_1)\}$.

Definition 3. Tensor product $G_1 \otimes G_2$: $V(G_1 \otimes G_2) = V_1 \times V_2 = \{(a, b) : a \in V_1 \ \wedge \ b \in V_2\}$ and $E(G_1 \otimes G_2) = \{((a, b), (a', b')) : a, a' \in V_1 \ \wedge \ b, b' \in V_2 \ \wedge \ (a, a') \in E_1 \ \wedge \ (b, b') \in E_2\}$.

Definition 4. Normal product $G_1 \star G_2$: $V(G_1 \star G_2) = V_1 \times V_2 = \{(a, b) : a \in V_1 \ \wedge \ b \in V_2\}$ and $E(G_1 \star G_2) = E(G_1 \times G_2) \ \cup \ E(G_1 \otimes G_2)$, where $E(G_1 \times G_2)$ and $E(G_1 \otimes G_2)$ are the edge–sets of the Cartesian and conjunctive products of G_1 and G_2 respectively.

The tensor product (also called the Kronecker product [20], categorical product [12] and conjunctive product) is one of the least understood graph products. The lexicographic product is also known as composition and was introduced by Sabidussi [13]. Note that of the four products, only the lexicographic product is not commutative.

We conclude this section by giving a few examples where the product of two graphs is A–magic, but the individual factors are not A–magic.

Example 1. Consider the graph $G = P_4 \times P_4$. Figure 1 shows that G is Z_k–magic, for $k \neq 2$. However, P_4 is not Z_k–magic, for any k.

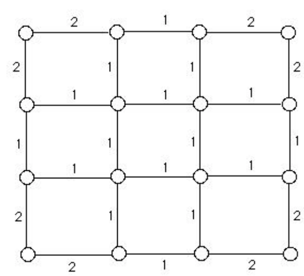

Figure 1. $G = P_4 \times P_4$.

Example 2. Consider the graph $G = P_4 \bullet N_2$, where N_2 is the null graph of order two (Figure 2). Since G is an eulerian graph with an even number of edges, we can label the edges of the eulerian circuit with $a, -a, a, -a, ..., a, -a$, where $a \in A^*$. Thus, G is A–magic. Clearly, P_4 and N_2 are not A–magic.

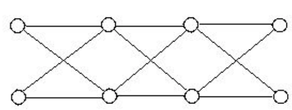

Figure 2. $G = P_4 \bullet N_2$.

Example 3. Consider the graph $G = P_4 \otimes P_4$. Figure 3 shows that G is Z_{2k+1}–magic, for all k. Clearly, P_4 is not Z_{2k+1}–magic.

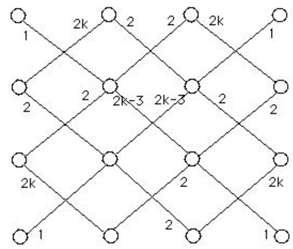

Figure 3. $G = P_4 \otimes P_4$.

3. Products of group–magic graphs

Let us now analyze the A–magic property of the resulting graph obtained from the product of two A–magic graphs. For A–magic graphs $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$, let L_1 and L_2 represent their respective A–magic labelings. Furthermore, let w_1 and w_2 be the constants induced on V_1 and V_2 respectively, by these labelings. Thus, we have $\sum_{a'} L_1(a, a') = w_1$ for any vertex $a \in V_1$ and $\sum_{b'} L_2(b, b') = w_2$ for any vertex $b \in V_2$.

To illustrate the theorems in this section, we will use the labeled graphs found in Figure 4.

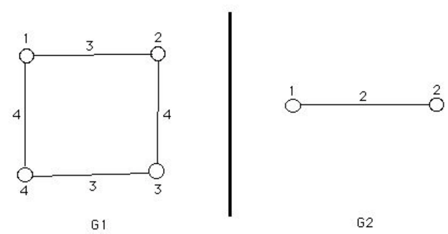

Figure 4. Z_7–magic labelings of G_1 and G_2.

Theorem 1. *Let A be an abelian group. If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are A–magic graphs, then the Cartesian product $G_1 \times G_2$ is A–magic.*

Proof. Let L denote the labeling assignment of $E(G_1 \times G_2)$, defined by:

$$L((a,b),(a',b')) = \begin{cases} L_1(a,a'), & \text{if } b = b'. \\ L_2(b,b'), & \text{if } a = a'. \end{cases}$$

Then, the induced labeling of every vertex (a, b) is:

$$\begin{aligned} \sum_{a',b'} L((a,b),(a',b')) &= \sum_{b'} L((a,b),(a,b')) + \sum_{a'} L((a,b),(a',b)) \\ &= \sum_{b'} L_2(b,b') + \sum_{a'} L_1(a,a') \\ &= w_2 + w_1. \end{aligned}$$

□

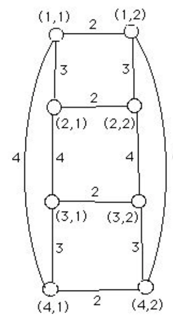

Figure 5. Z_7–magic labeling of the Cartesian product $G_1 \times G_2$.

Theorem 2. *Let A be an abelian group. If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are A–magic graphs, then both the lexicographic products $G_1 \bullet G_2$ and $G_2 \bullet G_1$ are A–magic.*

Proof. We will show that $G_1 \bullet G_2$ is A–magic. Let L denote the labeling assignment of $E(G_1 \bullet G_2)$, defined by:

$$L((a,b),(a',b')) = \begin{cases} L_2(b,b'), & \text{if } a = a'. \\ L_1(a,a'), & \text{otherwise.} \end{cases}$$

Then, the induced labeling of every vertex (a,b) is:

$$\begin{aligned} \sum_{a',b'} L((a,b),(a',b')) &= \sum_{\substack{a',b' \\ a=a'}} L((a,b),(a',b')) + \sum_{\substack{a',b' \\ a\neq a'}} L((a,b),(a',b')) \\ &= \sum_{b'} L_2(b,b') + \sum_{a'} \sum_{b'} L_1(a,a') \\ &= w_2 + \sum_{a'} \{p_2 \cdot L_1(a,a')\} \\ &= w_2 + p_2 \cdot w_1. \end{aligned}$$

A similar argument is used to show that $G_2 \bullet G_1$ is A–magic. □

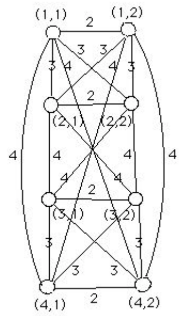

Figure 6. Z_7–magic labeling of the lexicographic product $G_1 \bullet G_2$.

Theorem 3. *Let A be an abelian group, underlying a commutative ring R. If there exist A–magic labelings $L_1 : E(G_1) \to A^* \cap U(R)$ and $L_2 : E(G_2) \to A^* \cap U(R)$ for graphs G_1 and G_2 respectively, then the tensor product $G_1 \otimes G_2$ is A–magic.*

Proof. Let L denote the labeling assignment of $E(G_1 \otimes G_2)$, defined by:

$$L((a,b),(a',b')) = L_1(a,a') \cdot L_2(b,b').$$

Then, the induced labeling of every vertex (a,b) is:

$$\begin{aligned}\sum_{a',b'} L((a,b),(a',b')) &= \sum_{a'} \sum_{b'} \{L_1(a,a') \cdot L_2(b,b')\} \\ &= \sum_{a'} L_1(a,a') \cdot \sum_{b'} L_2(b,b') \\ &= w_1 \cdot w_2.\end{aligned}$$

Note that L assigns non–zero elements to $E(G_1 \otimes G_2)$, since the range of L_1 and L_2 are subsets of $A^* \cap U(R)$. □

Corollary 1. *Let A be an abelian group, underlying a field F. If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are A–magic graphs, then the tensor product $G_1 \otimes G_2$ is A–magic.*

Proof. This is an immediate consequence of Theorem 3. □

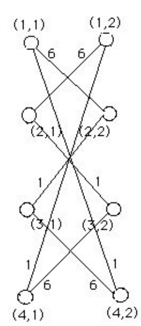

Figure 7. Z_7–magic labeling of the tensor product $G_1 \otimes G_2$.

Theorem 4. *Let A be an abelian group, underlying a commutative ring R. If there exist A–magic labelings $L_1 : E(G_1) \to A^* \cap U(R)$ and $L_2 : E(G_2) \to A^* \cap U(R)$ for graphs G_1 and G_2 respectively, then the normal product $G_1 \star G_2$ is A–magic.*

Proof. Note the following: $E(G_1 \star G_2) = E(G_1 \times G_2) \cup E(G_1 \otimes G_2)$ and $E(G_1 \times G_2) \cap E(G_1 \otimes G_2) = \emptyset$. Let L denote the labeling assignment of $E(G_1 \star G_2)$, defined by:

$$L((a,b),(a',b')) = \begin{cases} L_1(a,a'), & \text{if } b = b'. \\ L_2(b,b'), & \text{if } a = a'. \\ L_1(a,a') \cdot L_2(b,b'), & \text{otherwise.} \end{cases}$$

Then, the induced labeling of every vertex (a,b) is:

$$\begin{aligned} \sum_{a',b'} L((a,b),(a',b')) &= \sum_{b'} L((a,b),(a,b')) + \sum_{a'} L((a,b),(a',b)) \\ &+ \sum_{\substack{a' \\ a' \neq a}} \sum_{\substack{b' \\ b' \neq b}} L((a,b),(a',b')) \\ &= \sum_{b'} L_2(b,b') + \sum_{a'} L_1(a,a') \\ &+ \sum_{a'} L_1(a,a') \cdot \sum_{b'} L_2(b,b') \\ &= w_2 + w_1 + w_1 \cdot w_2. \end{aligned}$$

L assigns non–zero elements to $E(G_1 \star G_2)$, since the range of L_1 and L_2 are subsets of $A^* \cap U(R)$. □

Corollary 2. *Let A be an abelian group, underlying a field F. If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are A–magic graphs, then the normal product $G_1 \star G_2$ is A–magic.*

Proof. This is an immediate consequence of Theorem 4. □

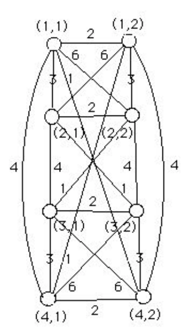

Figure 8. Z_7–magic labeling of the normal product $G_1 \star G_2$.

References

[1] M. Doob, *On the construction of magic graphs*, Proc. Fifth S.E. Conference on Combinatorics, Graph Theory and Computing (1974), 361–374.

[2] M. Doob, *Generalizations of magic graphs*, Journal of Combinatorial Theory, Series B, **17** (1974), 205–217.

[3] M. Doob, *Characterizations of regular magic graphs*, Journal of Combinatorial Theory, Series B, **25** (1978), 94–104.

[4] M.C. Kong, S–M Lee, and H. Sun, *On magic strength of graphs*, Ars Combinatoria, **45** (1997), 193–200.

[5] A. Kotzig and A. Rosa, *Magic valuations of finite graphs*, Canad. Math. Bull., **13** (1970), 451–461.

[6] S–M Lee, F. Saba, E. Salehi, and H. Sun, *On the V_4–group magic graphs*, Congressus Numerantium, **156** (2002), 59–67.

[7] S–M Lee, F. Saba, and G. C. Sun, *Magic strength of the k–th power of paths*, Congressus Numerantium, **92** (1993), 177–184.

[8] S–M Lee and E. Salehi, *Integer–magic spectra of amalgamations of stars and cycles*, Ars Combinatoria, **67** (2003), 199–212.

[9] S–M Lee, Hugo Sun, and Ixin Wen, *On group-magic graphs*, Journal of Combinatorial Mathematics and Combinatorial Computing, **38** (2001), 197–207.

[10] S–M Lee, L. Valdes, and Yong–Song Ho, *On integer–magic spectra of trees, double trees and abbreviated double trees*, Journal of Combinatorial Mathematics and Combinatorial Computing, **46** (2003), 85–96.

[11] R.M. Low and S–M Lee, *On group–magic eulerian graphs*, Journal of Combinatorial Mathematics and Combinatorial Computing, **50** (2004), 141–148.

[12] D.J. Miller, *The categorical product of graphs*, Canad. J. Math., **20** (1968), 1511–1521.

[13] G. Sabidussi, *The lexicographic product of graphs*, Duke Math. J., **28** (1961), 573–578.

[14] J. Sedlacek, *On magic graphs*, Math. Slov, **26** (1976), 329–335.

[15] J. Sedlacek, *Some properties of magic graphs*, in Graphs, Hypergraph, and Bloc Syst. 1976, Proc. Symp. Comb. Anal., Zielona Gora (1976), 247–253.

[16] R.P. Stanley, *Linear homogeneous diophantine equations and magic labelings of graphs*, Duke Math. J., **40** (1973), 607–632.

[17] R.P. Stanley, *Magic labeling of graphs, symmetric magic squares, systems of parameters and Cohen–Macaulay rings*, Duke Math. J., **40** (1976), 511–531.

[18] G.W. Sun and S–M Lee, *Construction of Magic Graphs*, Congressus Numerantium, **103** (1994), 243–251.

[19] W.D. Wallis, *Magic Graphs*, Birkhauser Boston, (2001).

[20] P.M. Weichsel, *The Kronecker product of graphs*, Proc. Amer. Math. Soc., **13** (1962), 47–52.

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