On (k,1)- Strongly Indexable Graphs Associated with Sequences of Positive Integers

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Dedicated to Professor S. M. Hegde

ABSTRACT

For any integers k ,d$\geq$ 1, a (p,q)-graph G with vertex set V(G) and edge set E(G), p=|V(G)| and q=|E(G)|, is said to be (k,d)-strongly indexable (in short **(k,d)-SI**) if there exists a function pair (f, f^+) which assigns integer labels to the vertices and edges, i.e., f: V (G) $\rightarrow$\{0,1,...,p-1\} and f^+: E (G) $\rightarrow$\{k,k+d, k+2d,...,k+(q-1)d\} are onto, where f^+(u, v) = f (u)+f (v) for any (u, v) $\in$ E(G). For any sequence of positive integers ($a_1,a_2,\dots,a_n$) with $a_i \geq 2$, for i =1,2,...,n-1, we associate some (1,1)-SI graphs. We contain the result of Acharya and Hegde [8] on level joined planar grids as special case.

1. Introduction. In 1990, Acharya and Hegde [2] have introduced the concept of strongly k-indexable graphs: A (p, q)-graph G = (V; E) with p vertices and q edges is said to be **strongly k-indexable** if its vertices can be assigned distinct numbers 0,1, 2, ...,p-1 so that the values of the edges, obtained as the sums of the numbers assigned to their end vertices form an arithmetic progression k, k + 1, k + 2,..., k + (q − 1). When k = 1 strongly k-indexable graph is simply called strongly indexable graph. Later, they extend the concept to the following definition

Definition 1.1. For any integers k ,d $\geq$ 1, a graph G with vertex set V(G) and edge set E(G), p=|V(G)| and q=|E(G)|, is said to be **(k, d)- strongly indexable** (in short **(k, d)-SI**) if there exists a function pair (f, f^+) which assigns integer labels to the vertices and edges as follows:

f: V (G) $\rightarrow$ \{0,1,...,p-1\} and f^+: E (G) $\rightarrow$\{k,k+d,k+2d,...,k+(q-1)d\} are onto, where f^+(u, v) = f (u)+f (v) for any (u, v) $\in$ E(G).

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Thus, the strongly k-indexable graphs are (k, 1)-strongly indexable and the strongly indexable graphs are (1, 1)-strongly indexable.

If we relaxed the definition of f in strongly (k, d)-indexable graph by $f: V(G) \rightarrow N$, then we have the concept of (k, d)-arithmetic graphs of Acharya and Hegde [1].

For any $k,d \geq 1$, we denote the class of all (k,d)-SI graphs by **Ω(k,d).**

Example 1. Figure 1 shows that the tree is (7,1)-, and (10,1)-SI.

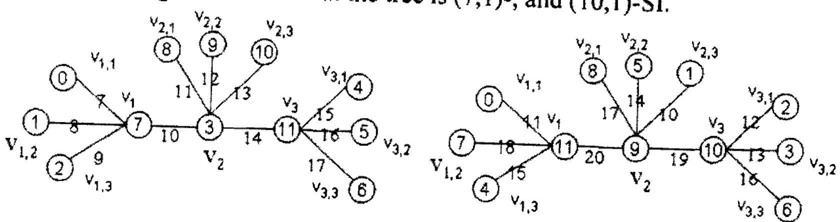

Figure 1. Tree which is (7,1)-SI and (10,1)-SI.

Example 2. The following are two different (1,1)-SI labeling of K_2xC_3.

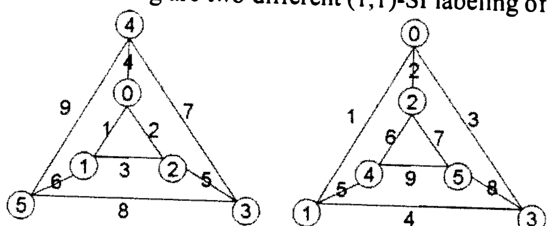

Figure 2. K_2xC_3 has different (1,1)-SI labeling.

Example 3. The following graphs are (1,1)-SI.

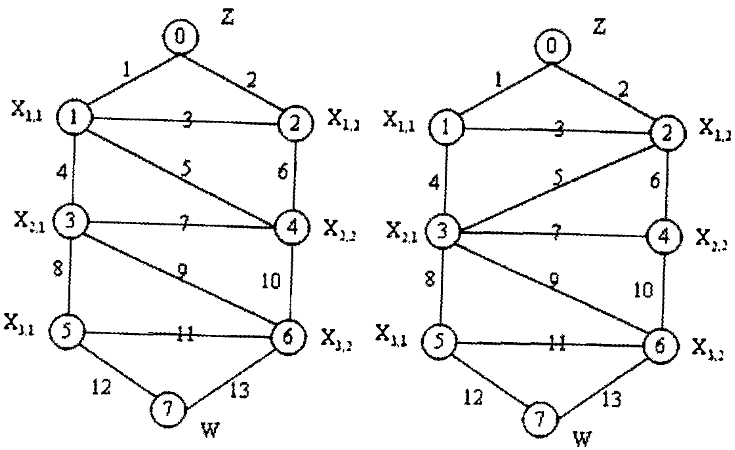

Figure 3. Two graphs with their (1,1)-SI labelings.

Acharya and Hegde showed that the only non-trivial regular graphs that are strongly indexable are K_2, K_3 and K_2 x K_3.and that every strongly indexable graph has exactly one non-trivial component that is either a star or a triangle. Results on strongly indexable graphs are meager. There are few examples of strongly indexable graphs were known.

General (k, d)-SI graphs were considered by the authors in [11]. Lee et al [14] determine classes of graphs that are (1, 2)-SI and (2, 2)-SI. Lee and Lo [13] determine classes of spiders that are (1,2)-SI.

For any sequence of positive integers $(a_1,a_2,\ldots,a_n)$ with $a_i \geq 2$, for $i = 1,2, \ldots,n$, we associate (1,1)-SI graph $H(<a_1,a_2,\ldots,a_n>)$. We obtain the result of Acharya and Hegde [8] on level joined planar grids as special case.

2. A Construction of (k,d)-SI graphs.

We have shown in [11] a general construction of (k, d)-SI graph from two given (k, d)-SI graphs.

Ingredient: Suppose G is a (p_1,q_1)-graph in $\Omega(k_1,d)$ and H is a (p_2,q_2)-graph in $\Omega(k_2,d)$ with labelings g,h respectively.

Constraint: d is a divisor of $2p_1+(k_2-k_1)$ and $[2p_1+(k_2-k_1)]/d - q_1 \geq 0$.

We can construct a new graph on $V(G) \cup V(H)$ as follows:

Keep the original (k_1,d)-labeling on G and extend the vertex labeling on H by $h\oplus p_1$ where $(h\oplus p_1)(v) = h(v)+p_1$ for all $v \in V(H)$.

Under the $h\oplus p_1$ labeling H becomes a $(2p_1+k_2,d)$-SI graphs.

Let $t = [2p_1+(k_2-k_1)]/d - q_1 \geq 0$.

If $t=0$, then the disjoint union $G \cup H$ is (k_1, d)-SI.

If $t>0$, let us fill in t edges which connect vertices of G and H by the following scheme:

Pick u in G with label x and v in H with label $2p_1+y$ join them so that its induced edge label $2p_1+x+y$ is range from k_1+q_1d to $k_1+(q_1+1)d, \ldots, k_1+(q_1+t-1)d$. We denote the set of these edges by Π. That is $\Pi = \{(u,v): g(u)=x$ and $h(v)=y$ and $x+y= k_1+q_1d, k_1+(q_1+1)d, \ldots, k_1+(q_1+t-1)d\}$.

Then $E(G) \cup E(H) \cup \Pi$ is (k_1,d)-SI.

We denote this graph by $G\oplus\Pi\oplus H$.

Theorem 2.1. If G is a (p_1,q_1)-graph in $\Omega(k_1,d)$ and H is a (p_2,q_2)-graph in $\Omega(k_2,d)$ and d is a factor of $2p_1+(k_2-k_1)$ with $[2p_1+(k_2-k_1)]/d - q_1 \geq 0$, then there exists a $(p_1+p_2, q_2+[2p_1+(k_2-k_1)]/d)$ graph in $\Omega(k_1,d)$ which contains G, H as induced subgraphs.

Theorem 2.2. If G is a (p_1,q_1)-graph and H is a (p_2,q_2)-graph in $\Omega(k,d)$ with $2p_1/d - q_1$ then the graph $G\oplus\Pi\oplus H$ is a $(p_1+p_2, 2p_1/d+q_2)$-graph in $\Omega(k,d)$.

We illustrate the above method by the following examples:

Example 4. Let $G = P_2$ and $H = C_3$. We see that $p_1=2$ and $q_1=1$. G,H are (1,1)-SI. Thus $t=2p_1/d - q_1 = 4-1=3$. Hence we need to add three edges with edge labels 2,3,4. Since $2 = 0+2$, $3 = 0+3 = 1+2$ and $4 = 0+4 = 1+3$. Thus we have $\Pi_1 = \{(0,2),(1,2),(0,4)\}$ or $\Pi_2 = \{(0,2),(0,3),(0,4)\}$.

Figure 4 depicts $G\oplus\Pi_1\oplus H$ and $G\oplus\Pi_2\oplus H$.

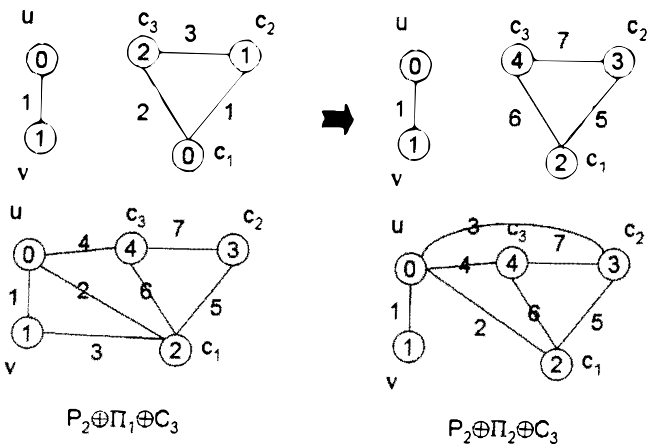

Figure 4. The graphs depicts $G\oplus\Pi_1\oplus H$ and $G\oplus\Pi_2\oplus H$ where $G = P_2$ and $H = C_3$.

Example 5. Consider $G = P_7$, $H = P_5$ which are (1,2)-SI. Thus $t = [2p_1 + (k_2-k_1)]/d - q_1 = [2x7+(1-1)]/2 - (7-1) = 1$. Hence $\Pi = \{(x_{1,3}, y_{1,5})\}$. We see that $G\oplus\Pi\oplus H$ is (1,2)-SI.

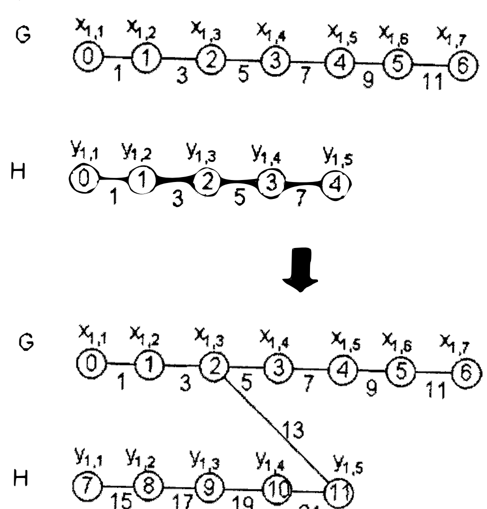

Figure 5. The graphs depicts $P_7\oplus\Pi\oplus P_5$.

Example 6. Let $G = P_9$ and $H = P_6$ with the (1,2)-SI labeling as shown in Figure 6. As $t = [2p_1 + (k_2-k_1)]/d - q_1 = [2x9+(1-1)]/2 - (9-1) = 1$.

Hence $\Pi = \{(x_{1,5}, y_{1,1})\}$. We see that $G\oplus\Pi\oplus H$ is (1,2)-SI.

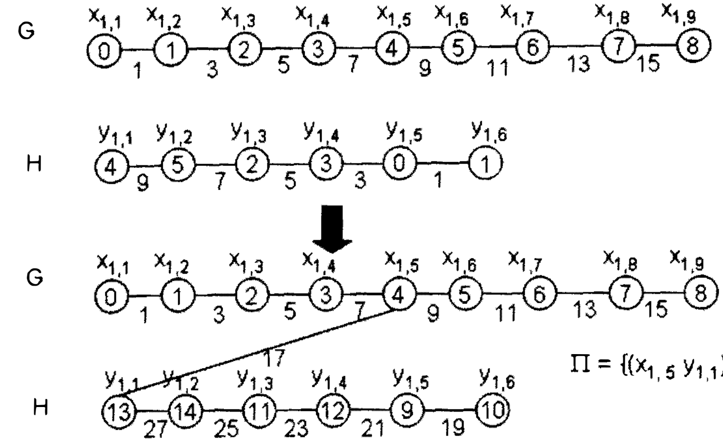

Figure 6. The graphs depicts $P_9\oplus\Pi\oplus P_6$.

3. (k,1)- Strongly Indexable Graphs Associated with Sequences of Positive Integers

We illustrate here the usefulness of this method by presenting a recursive construction of infinite families of (k,1)-SI graphs associate with sequences of positive integers.

Denote $N^{>1} = \{2,3,4,\ldots\}$. Let σ be a sequence of non-zero integers $<n_1, n_2,\ldots, n_t>$ of $N^{>1}$ where $t \geq 2$. If $n_1 = k \geq 2$, we denote PSG(t,k) the class of (k,1)-SI graphs constructed by the following way:

Initiate case t =2.

PSG(2,k) consists of graphs of the form $H(<n_1, n_2>)$ where $n_2 \in N^{>1}$

$V\,H(<n_1, n_2>) = \{ v(1,1),\ldots v(1,n_1), v(2,1),\ldots v(2,n_2)\}$

$H(<n_1, n_2>)$ is with $n_1 + n_2$ vertices with two layers. The topmost layer has $v(1,1),\ldots v(1,n_1)$ and the second layer has vertices $v(2,1),\ldots v(2,n_2)$.

$E\,H(<n_1, n_2>)$ has edges in the form:

(1) If $n_2 \leq n_1$, then

$E\,H(<n_1, n_2>) = \{(v(1,i), v(2,i)), (v(1,i+1), v(2,i)): i= 1,2,\ldots, n_2\} \cup \{(v(1,i+1), v(2, n_1)): i= n_2,\ldots, n_1-1\} \cup \{(v(2, 1), v(2, n_2))\}$

(2) If $n_1 \leq n_2$, then

$E\,H(<n_1, n_2>) = \{(v(1,i), v(2,i)), (v(1,i+1), v(2,i)): i= 1,2,\ldots, n_1\} \cup \{(v(1, n_1), v(2, i+1)): i= n_1,\ldots, n_2-1\} \cup \{(v(2, 1), v(2, n_2))\}$.

It is (k,1)-SI with the following labeling:

$f(v(1,i)) = i$ for $i = 1,\ldots, n_1$ and $f(v(2,i)) = n_1 + i - 1$ for $i = 1,\ldots, n_2$ (see Figure 7 for $H(<5,3>)$ and $H(<5,8>)$).

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PSG(2,k) consists of graphs of the form $H(<n_1, n_2>)$ where $n_2 \in N^{>1}$

$V\,H(<n_1, n_2>) = \{ v(1,1),\ldots v(1,n_1), v(2,1),\ldots v(2,n_2)\}$

$H(\langle n_1, n_2\rangle)$ is with $n_1 + n_2$ vertices with two layers. The topmost layer has $v(1,1),\ldots v(1,n_1)$ and the second layer has vertices $v(2,1),\ldots v(2,n_2)$.

$E\,H(\langle n_1, n_2\rangle)$ has edges in the form:

(1) If $n_2 \le n_1$, then

$$E\,H(\langle n_1, n_2\rangle)=\{(v(1,i), v(2,i)), (v(1,i+1), v(2,i)): i= 1,2,\ldots, n_2\}\cup \{(v(1,i+1), v(2, n_1)): i= n_2,\ldots, n_1-1\}\cup \{(v(2, 1), v(2, n_2))\}$$

(2) If $n_1 \le n_2$, then

$$E\,H(\langle n_1, n_2\rangle)=\{(v(1,i), v(2,i)), (v(1,i+1), v(2,i)): i= 1,2,\ldots, n_1\}\cup \{(v(1, n_1), v(2, i+1)): i= n_1,\ldots, n_2-1\}\cup \{(v(2, 1), v(2, n_2))\}.$$

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$f(v(1,i)) = i$ for $i =1,\ldots, n_1$ and $f(v(2,i)) = n_1+i-1$ for $i =1,\ldots, n_2$ (see Figure 7 for $H(\langle 5,3\rangle)$ and $H(\langle 5,8\rangle)$).

Example 7.

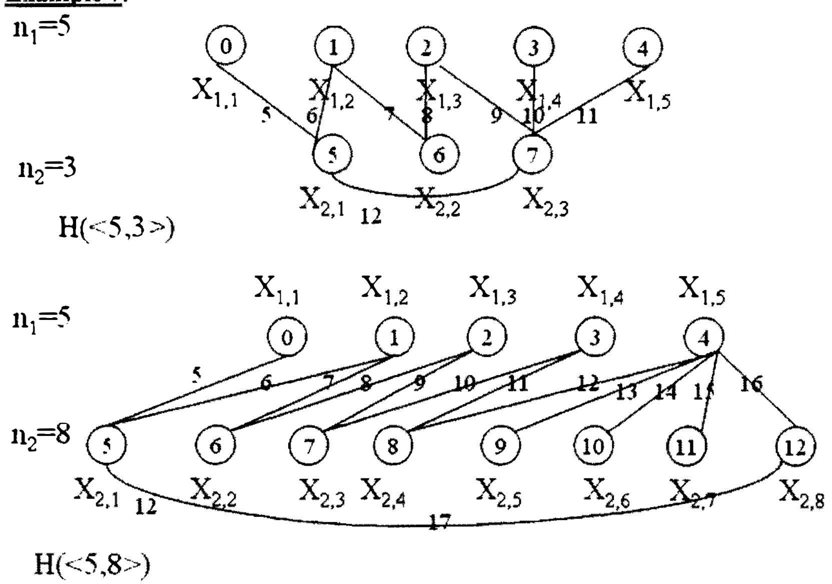


Figure 7. $H(\langle 5,3\rangle)$ and $H(\langle 5,8\rangle)$.

Induction Hypothesis. Assume we have $G = H(\langle n_1, n_2,\ldots,n_t\rangle)$ in $PSG(t,k)$ which is (k,1)-SI where $k = n_1$.

For any $n_{t+1} \in N^{>1}$.

Let W be the graph with $V(W) = v(t+1,1), \ldots, v(t+1,n_{t+1})\}$ and

$$E(W) = \{(v(t+1,1), v(t+1,n_{t+1}))\}$$

We can construct a new graph $H(\langle n_1, n_2,\ldots,n_t, n_{t+1}\rangle)$ as follows:

The graph $H(\langle n_1, n_2,\ldots,n_t, n_{t+1}\rangle)$ has $n_1+ n_2+\ldots+ n_k +n_{t+1}$ vertices.

$$V(H(\langle n_1, n_2,\ldots,n_t, n_{t+1}\rangle)) =V(H(\langle n_1, n_2,\ldots,n_t\rangle))\cup\{ v(t+1,1), \ldots,v(t+1,n_{t+1})\}.$$

We arrange the vertices of $H(\langle n_1, n_2,\ldots,n_t, n_{t+1}\rangle))$ layer by layer and from left to right as follows:

(A) the topmost layer is $H(\langle n_1, n_2,\ldots,n_t\rangle)$,

(B) the second layer has vertices $v(t+1,1),\ldots v(t+1, n_{t+1})$,

$H(\langle n_1, n_2,\ldots,n_t, n_{t+1}\rangle)$ has edges in the form:

(1) If $n_{t+1} \le n_t$, then

$$EH(\langle n_1, n_2,\ldots,n_t, n_{t+1}\rangle) = EH(\langle n_1, n_2,\ldots,n_t\rangle) \cup E(W) \cup \{(v(t,i), v(t+1,i)), (v(t,i+1), v(t+1,i)): i= 1,2,\ldots, n_{t+1}\}\cup \{(v(t,i+1), v(t+1, n_t)): i= n_{t+1},\ldots, n_t-1\}$$

(2) If $n_t \le n_{t+1}$, then

$$EH(\langle n_1, n_2,\ldots,n_t, n_{t+1}\rangle) = EH(\langle n_1, n_2,\ldots,n_t\rangle) \cup E(W) \cup \{(v(t,i), v(t+1,i)), (v(t,i+1), v(t+1,i)): i= 1,2,\ldots, n_t\}\cup \{(v(t, n_t), v(t+1, i+1)): i= n_t,\ldots, n_{t+1}-1\}.$$

We can extend the labeling of f in $G = H(\langle n_1, n_2,\ldots,n_t\rangle)$ to $H(\langle n_1, n_2,\ldots,n_t, n_{t+1}\rangle)$ by defining
$f(v(t+1,i)) = n_1+ n_2+\ldots+n_t+ i -1$ for $i =1,\ldots, n_{t+1}$

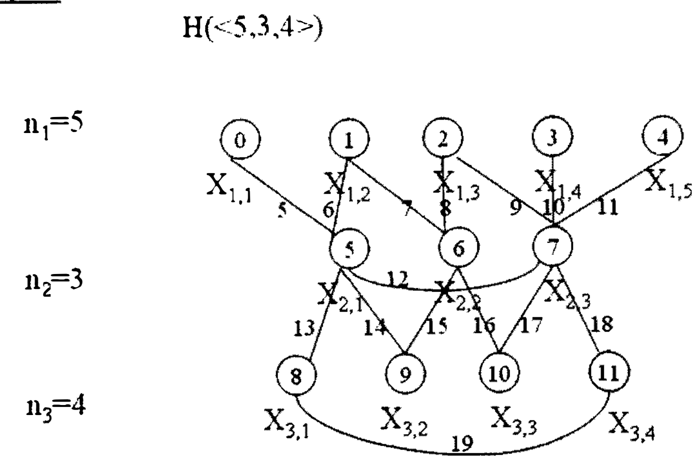
Theorem 3.1. The graph $H(\langle n_1, n_2,\ldots,n_t, n_{t+1}\rangle)$ is is (k,1)-SI where $k = n_1$.

Proof. In fact, $H(\langle n_1, n_2,\ldots,n_t, n_{t+1}\rangle)$ is $G\oplus\Pi\oplus H$ of Theorem 2.2. where $G = H(\langle n_1, n_2,\ldots,n_t\rangle)$, $H = W$ and

(1) $\Pi=\{(v(t,i), v(t+1,i)), (v(t,i+1), v(t+1,i)): i= 1,2,\ldots, n_{t+1}\}\cup \{(v(t,i+1), v(t+1, n_t)): i= n_{t+1},\ldots, n_t-1\}$, if $n_{t+1} \le n_t$.

(2) $\Pi= \{(v(t,i), v(t+1,i)), (v(t,i+1), v(t+1,i)): i= 1,2,\ldots, n_t\}\cup \{(v(t, n_t), v(t+1, i+1)): i= n_t,\ldots, n_{t+1}-1\}$, if $n_t \le n_{t+1}$. □

Example 8.



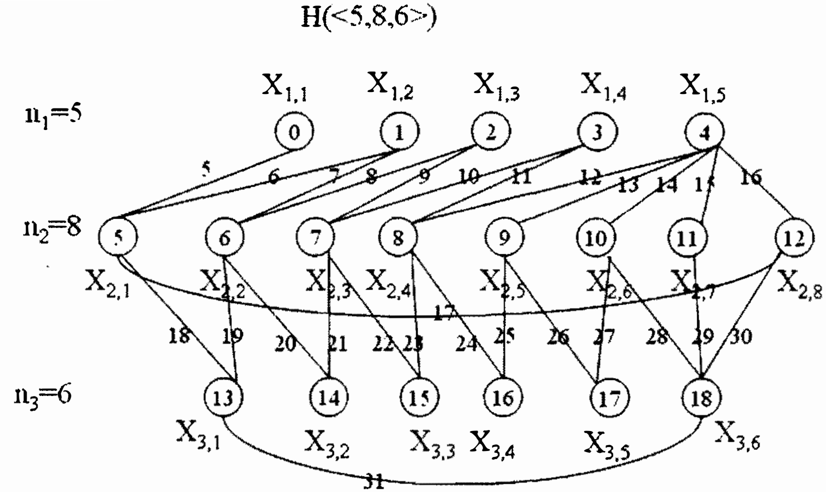


Figure 8. H(<5, 3, 4>) and H(<5, 8, 6>).

Now, we want to consider sequence σ on $N^{>0}$ ={1,2,3,4,...} with the following property: $\sigma =<1, n_1, n_2,\ldots,n_t, >$ where $n_1, n_2,\ldots,n_t$ in $N^{>1}$.

We can construct a new graph $H^*(<n_1, n_2,\ldots,n_t>)$ as follows:

The graph $H (<n_1, n_2,\ldots,n_t, >)$ has $n_1+ n_2+\ldots+ n_t$ vertices is in PSG(t,k).

$V(H^*(<n_1, n_2,\ldots,n_t>)) =V(H(<n_1, n_2,\ldots,n_t>))\cup\{z\}$.

We arrange the vertices of $H (<n_1, n_2,\ldots,n_t>))$ layer by layer and from left to right as before. The lower layer has vertex z.

$H^*(<n_1, n_2,\ldots,n_t>)$ has edges in the form:

$E(H^*(<n_1, n_2,\ldots,n_t>))= E(H(<n_1, n_2,\ldots,n_t>)) \cup \{(v(t,i), z)): i= 1,2,\ldots, n_t\}$

It is follows that

Theorem 3.2. The graph $H^*(<n_1, n_2,\ldots,n_t>)$ is $(n_1,1)$-SI.

Example 9. Figure 9 depicts H*(<5,3>).

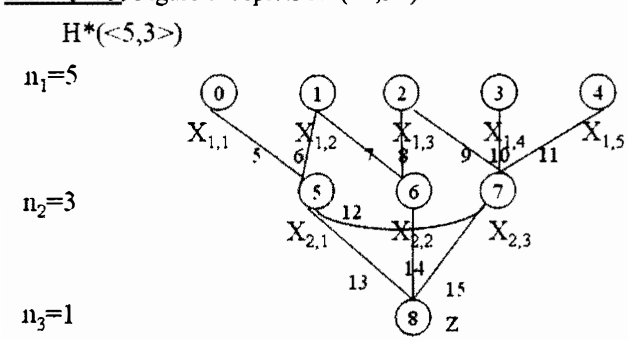


Figure 9.

Dually, we can construct a new graph $^*H (<n_1, n_2,\ldots,n_t>)$ as follows:

The graph $H(<n_1, n_2,\ldots,n_t, >)$ has $n_1+ n_2+\ldots+ n_k +n_{t+1}$ vertices is in PSG(t,k).

$V(H(<n_1, n_2,\ldots,n_t>)) = \{u\}\cup V(H(<n_1, n_2,\ldots,n_t>))$

The upper layer has vertex u. We arrange the vertices of $H(<n_1, n_2,\ldots,n_t, n_{t+1}>))$ layer by layer and from left to right as before.

$^*H (<n_1, n_2,\ldots,n_t>)$ has edges in the form:

$E(H^*(<n_1, n_2,\ldots,n_t>))= \{(u, v(1,i))): i= 1,2,\ldots, n_1\}\cup E(H(<n_1, n_2,\ldots,n_t>))$

It is follows that

Theorem 3.3. The graph $^*H(<n_1, n_2,\ldots,n_t>)$ is (1,1)-SI.

Example 10. We illustrate *H<3, 3>, *H<4, 4> in Figure 10.

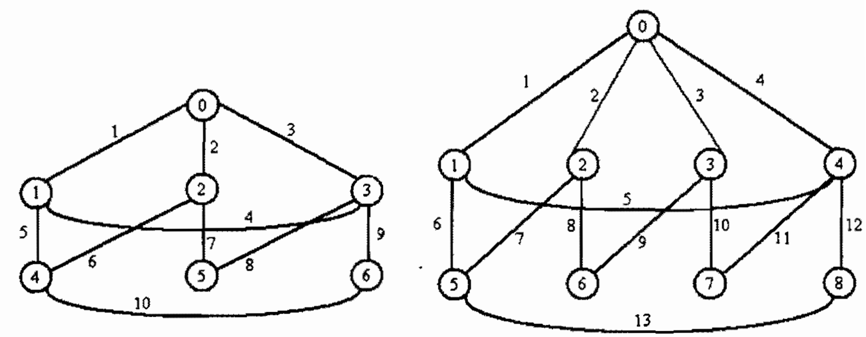


Figure 10. *H<3, 3> and *H<4, 4>.

Example 11. We illustrate *H<2,3,2>,*H<2,3,4,3> in Figure 11.

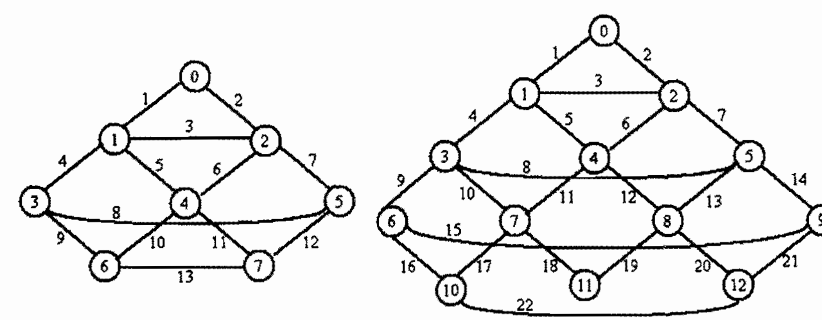


Figure 11. *H<2,3,2> and *H<2,3,4,3>.

Hegde and Shetty [8] introduced a class of strongly indexable graphs which they called ***level joined planar grids*:** Let u be a vertex of $P_m\times P_n$ such that $deg\ (u) = 2$. Introduce an edge between every pair of distinct vertices v, w with $deg\ (v)$, $deg\ (w) \neq 4$ if $d(u, v) = d(u, w)$ where $d(u, v)$ is the distance between u and v. The graph so obtained is defined as the *level joined planar grid* and is denoted by LJ: $P_m\times P_n$. An example LJ: $P_4\times P_5$ is illustrated in Figure 12.

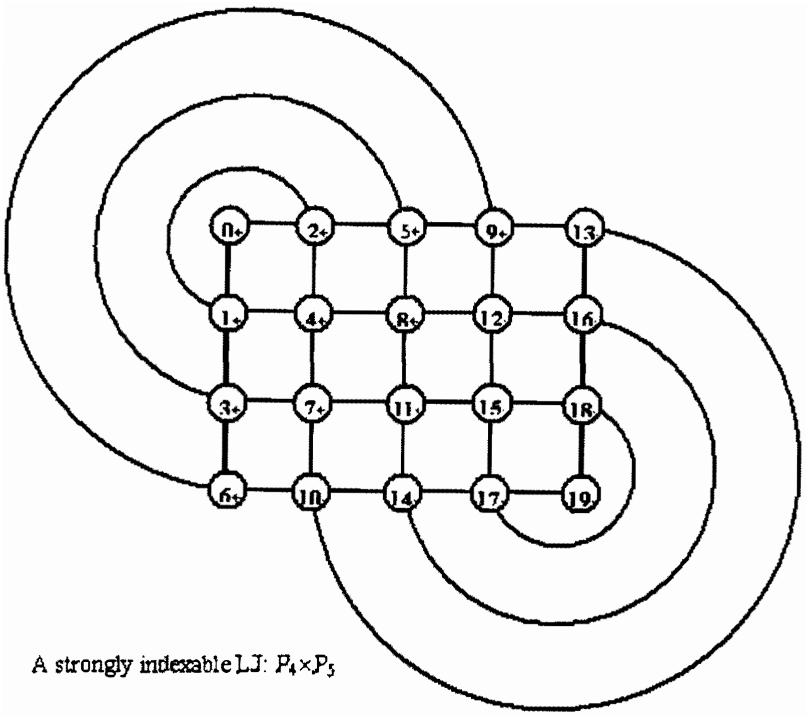


Figure 12. LJ: $P_4 x P_5$.

We observe that LJ: $P_m \times P_n$ is the graph *H*<2,3,4,…,n-1,(n-m)'s n,n-1,n-2,…,3, 2>. The Figure 18 is m=4 and n=5, it is *H*<2,3,4,4,3,2>.

Hence by Theorem 3.3, we have

Corollary 3.4. The graph *LJ*: $P_m \times P_n$ is (1,1)-SI.

4. Final Remark.

A (p,q)-graph G= (V, E) with p vertices and q edges is called ***total edge magic*** if there is a bijection f : V∪ E–>{1,2,...,p+q} such that there exists a constant s for any (u,v) in E f(u) + f((u,v)) + f(v) = s. The original concept of a total edge-magic graph is credited to Kotzig and Rosa [9,10]. Originally, they termed it as a magic graph. Motivated by the definition of total edge-magic labelings, Enomoto,Llado, Nakamigawa and Ringel [4] introduced the concept of super edge-magic graphs in 1998. A total edge-magic graph is called ***super edge-magic*** if f(V(G))= {1,2,…,p}.

Super edge-magic graphs were studied in [3,4,8,12]. Recently, Hegde and S. Shetty [8] showed that the concepts of (1,1)-SI and super edge-magic are equivalent. Thus our paper produces infinitely many super edge-magic graphs.

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