On Edge-Balance Index Sets of Generalized Theta Graphs

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Abstract

Any edge labeling $f : E \to \{0,1\}$ of a simple graph $G = (V, E)$ induces a vertex labeling $f^* : V \to \{0,1\}$ defined by $f^*(x) = i$ if x is incident to more i-edges than $(1-i)$-edges, and $f^*(x)$ is unlabeled if x is incident to an equal number of 0- and 1-edges. Denote by $e_f(i)$ and $v_f(i)$ the number of edges and vertices, respectively, labeled i. We call f edge-friendly if $|e_f(0) - e_f(1)| \leq 1$. Define the edge-balance index set of G as $\{\,|v_f(0) - v_f(1)| :$ the edge labeling f is edge-friendly$\}$. We study the edge-balance index sets of generalized theta graphs.

1 Introduction

We propose a new problem in graph labeling. It can be viewed as a dual version of balance index sets, a research topic that has produced many interesting results recently [2]–[13]. Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. Given any edge labeling $f : E \to \{0,1\}$, we define an associated partial vertex labeling $f^* : V \to \{0,1\}$ as follows. Define $f^*(v)$ to be 0 if it is incident to more 0-edges than 1-edges, and 1 if it is incident to more 1-edges than 0-edges. If the vertex v is incident to an equal number of 0- and 1-edges, leave it unlabeled. Hence f^* is a partial function. For each $i \in \{0,1\}$. let $e_f(i) = |\{uv \in E : f(uv) = i\}|$, and $v_f(i) = |\{v \in V : f^*(v) = i\}|$. If no ambiguity occurs, we could omit the subscript and simply write $e(i)$ and $v(i)$ respectively.

Definition 1.1. A graph G is said to be ***edge-balanced*** if it admits an edge labeling f such that $|e_f(0) - e_f(1)| \leq 1$ and $|v_f(0) - v_f(1)| \leq 1$.

Requiring both $|e_f(0) - e_f(1)| \leq 1$ and $|v_f(0) - v_f(1)| \leq 1$ is difficult to meet. We could relax one of the two restrictions.

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Definition 1.2. A graph G is said to be ***edge-friendly graph*** if it admits an edge labeling f such that $|e_f(0) - e_f(1)| \leq 1$.

Obviously, an edge-friendly graph needs not be edge-balanced. To investigate how close it is to being edge-balanced, we introduce the following notion.

Definition 1.3. The ***edge-balance index set*** (or simply ***EBI set***) of a graph G is defined as

$$\mathrm{EBI}(G) = \{|v_f(0) - v_f(1)| : \text{the edge labeling } f \text{ is edge-friendly}\}.$$

In other words, $\mathrm{EBI}(G)$ is the set of values that $|v_f(0) - v_f(1)|$ could attain as we go over all edge-friendly labelings of G.

Example 1. We find

$$\mathrm{EBI}(nK_2) = \begin{cases} \{0\} & \text{if } n \text{ is even,} \\ \{2\} & \text{if } n \text{ is odd.} \end{cases}$$

Why? If an edge is labeled 0, both its endpoints are labeled 0. If an edge is labeled 1, both its endpoints are labeled 1. Therefore, if n is even, there is an equal number of 0- and 1-edges, hence an equal number of 0- and 1-vertices. Likewise, if n is odd, the numbers of 0- and 1-edges differ by 1, hence the numbers of 0- and 1-vertices differ by 2. □

Example 2. The star $\mathrm{St}(n)$ is the tree with n pendant edges incident to a vertex (the center). We claim that

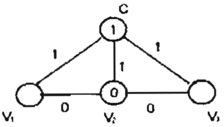
$$\mathrm{EBI}(\mathrm{St}(n)) = \begin{cases} \{0\} & \text{if } n \text{ is even,} \\ \{2\} & \text{if } n \text{ is odd.} \end{cases}$$

Here is the reason. Each of the n pendant vertices is labeled the same way as the edge incident to it. The center is either unlabeled, labeled 0 or 1, depending on whether $e(0) - e(1)$ equals 0, 1, or -1, respectively.

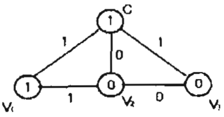
If n is even, we have $e(0) = e(1)$. Hence we have an equal number of 0- and 1-vertices among the end vertices. Along with the center, which is unlabeled, we find $v(0) - v(1) = 0$.

If $n = 2t + 1$, then $e(0) - e(1) = \pm 1$. If $e(0) - e(1) = 1$, the center is an 0-vertex. Among the $2t + 1$ pendant vertices, $t + 1$ of them are 0-vertices, the other t are 1-vertices. Therefore $v(0) - v(1) = 2$. Similarly, if $e(0) - e(1) = -1$, we find $v(0) - v(1) = -2$. □

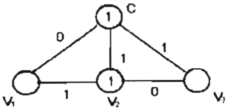
Example 3. The ***fan graph*** $F_{1,n}$ connects a vertex c to every vertex on a path of order n. Chopra, Lee and Su [1] showed that $\mathrm{EBI}(F_{1,3}) = \mathrm{EBI}(F_{1,4}) = \{0, 1, 2\}$. The labelings are displayed on the next page. □



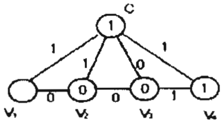
| v(0) - v(1) | = 0



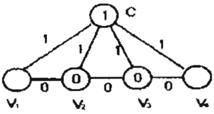
| v(0) - v(1) | = 1



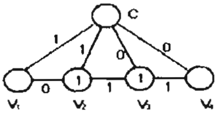
| v(0) - v(1) | = 2



| v(0) - v(1) | = 0

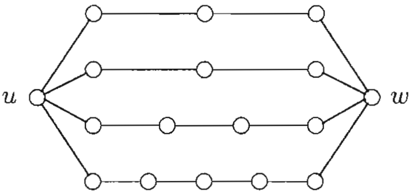


| v(0) - v(1) | = 1



| v(0) - v(1) | = 2

Take k paths of length $\ell_1, \ell_2, \ldots, \ell_k$, where $k \geq 3$ and $\ell_i = 1$ for at most one i. Identify their endpoints to form a new graph. The new graph is called a ***generalized theta graph***, and is denoted $\Theta(\ell_1, \ell_2, \ldots, \ell_k)$. In other words, $\Theta(\ell_1, \ell_2, \ldots, \ell_k)$ consists $k \geq 3$ pairwise internally disjoint paths of length $\ell_1, \ell_2, \ldots, \ell_k$ that share a pair of common endpoints u and w. Here is the generalized theta graph $\Theta(4,4,5,6)$:



In this paper, we study the edge-balance index sets of generalized theta graphs. For brevity, when $\ell_1 = \ell_2 = \cdots = \ell_k = a$, we will write $\Theta(a^{[k]})$.

2 EBI Sets of $\Theta(2^{[m]})$ and $\Theta(1, 2^{[m]})$

Theorem 2.1 *For any $m \geq 2$, EBI$(\Theta(2^{[m]})) = \{0\}$.*

Proof. Let x and y denote the numbers of *interior* vertices that are incident to two 0-edges and two 1-edges, respectively. Thus far, we have found $2x$ 0-edges, and $2y$ 1-edges. Half of the remaining $2m - 2x - 2y$ edges are 0-edges, half are 1-edges. Then $e(0) = m + x - y$, and $e(1) = m - x + y$. Thus $e(0) - e(1) = 2(x - y)$. Because of edge-friendliness, we need $x = y$. This means there is an equal number of 0- and 1-vertices among the m interior vertices. Hence *the value of $v(0) - v(1)$ depends solely on $f^*(u)$ and $f^*(w)$.* In general, however, we may need to adjust the result by a small value.

Let r and s be the number of 0-edges among the m copies of P_2 that are incident to u and w respectively. Then the number of 1-edges incident to u and w would be $m - r$ and $m - s$ respectively. Hence $e(0) = r + s$, $e(1) = 2m - (r + s)$, and $e(0) - e(1) = 2(r + s - m)$. To maintain edge-friendliness, we need $r + s = m$. Thus the two endpoints u and w are either both unlabeled, or they must be labeled differently. In either case, $\mathrm{EBI}(\Theta(2^{[m]})) = \{0\}$. □

Theorem 2.2 *For any* $m \geq 2$,

$$EBI(\Theta(1, 2^{[m]})) = \begin{cases} \{0, 1, 2\} & \textit{if } m \textit{ is even,} \\ \{0, 1\} & \textit{if } m \textit{ is odd.} \end{cases}$$

Proof. Compared to $\Theta(2^{[m]})$, the generalized theta graph $\Theta(1, 2^{[m]})$ has an extra edge uw. Notice that after changing all the 0-edges into 1-edges, and 1-edges into 0-edges, the new labeling is still edge-friendly. But an original 0-vertex becomes an 1-vertex, an 0-vertex turns into an 1-vertex, and any unlabeled vertex will remain unlabeled. Therefore the value of $|v(0) - v(1)|$ remains unchanged. Hence we may *assume uw is an 0-edge.*

Adopting the same notations used in the proof of Theorem 2.1, we find $e(0) - e(1) = 2(x - y) + 1$. Edge-friendliness implies $x - y = 0, -1$. We also have $e(0) - e(1) = 2(r + s - m) + 1$, thus $r + s = m, m - 1$.

If $r + s = m$, the numbers of 0- and 1-edges incident to u are $r + 1$ and s respectively, and the numbers of 0- and 1-edges incident to w are $s + 1$ and r respectively. Then we have

	$f^*(u)$	$f^*(w)$	$v(0) - v(1)$
$r \geq s + 2$	0	1	$x - y + 0$
$r = s + 1$	0	–	$x - y + 1$
$r = s$	0	0	$x - y + 2$
$r = s - 1$	–	0	$x - y + 1$
$r \leq s - 2$	1	0	$x - y + 0$

Notice that $r = s$ is only possible when $m = r + s = 2r$ is even.

If $r + s = m - 1$, then there are $r + 1$ and $s + 1$, respectively, 0- and 1-edges incident to u, and $s + 1$ and $r + 1$, respectively, 0- and 1-edges incident to w. We find

	$f^*(u)$	$f^*(w)$	$v(0) - v(1)$
$r < s$	1	0	$x - y + 0$
$r = s$	–	–	$x - y + 0$
$r > s$	0	1	$x - y + 0$

Since $x - y$ equals 0 or -1, the value of $|v(0) - v(1)|$ belongs to $\{0, 1, 2\}$ if m is even, but $\{0, 1\}$ if m is odd. It is easy to show that each of these values is attainable. This completes the proof. □

For $\Theta(2^{[m]},3))$, we may assume the three edges on P_3 are labeled 000, 001, 010, or 011. After a lengthy analysis similar to that used in the proof of Theorem 2.2, we find that $\Theta(2^{[m]},3)$ and $\Theta(1,2^{[m]})$ share the same EBI set. A shorter proof, however, can be found in Section 5.

It is clear that when $\ell_i > 2$ for some i, the analysis becomes more complicated. We need a better understanding of the combined effect of the individual labeling of the k paths on the entire graph. To achieve this goal, we first study the labeling (which needs not be edge-friendly) of paths.

3 EBI Sets of Paths and Cycles

Classify a labeling according to the first and the last edge labels (starting from u to w):

$$\begin{array}{ll} \text{type } 00: & 0\ldots0 \\ \text{type } 01: & 0\ldots1 \\ \text{type } 10: & 1\ldots0 \\ \text{type } 11: & 1\ldots1 \end{array}$$

A type 01 labeling starts with c_1 0-edges, d_1 1-edges, then c_2 0-edges, d_2 1-edges, and so forth, until it ends with c_b 0-edges and d_b 1-edges:

$$\text{edge labels}: \quad \underbrace{0\ldots0}_{c_1}\,\underbrace{1\ldots1}_{d_1}\,\underbrace{0\ldots0}_{c_2}\,\underbrace{1\ldots1}_{d_2}\quad\cdots\quad\underbrace{0\ldots0}_{c_b}\,\underbrace{1\ldots1}_{d_b}.$$

Hence the path consists of b blocks, where $b \geq 1$, such that each block consists of consecutive 0-edges followed by consecutive 1-edges.

Restricted to the ith block, where $1 \leq i \leq b$, it is easy to verify that

$$v(0) = \begin{cases} c_1 & \text{if } i = 1, \\ c_i - 1 & \text{if } i > 1, \end{cases} \qquad \text{and} \qquad v(1) = \begin{cases} d_i - 1 & \text{if } i < b, \\ d_b & \text{if } i = b. \end{cases}$$

Thus $v(0) = e(0) - (b-1)$, and $v(1) = e(1) - (b-1)$. Consequently, $v(0) - v(1) = e(0) - e(1)$ for any type 01 labeling. Note that *it does not depend on how the interior edges are labeled.*

Proceeding in a similar fashion, we find

$$v(0) - v(1) = \begin{cases} e(0) - e(1) & \text{type 01 and 10}, \\ e(0) - e(1) + 1 & \text{type 00}, \\ e(0) - e(1) - 1 & \text{type 11}. \end{cases}$$

As we have remarked in the proof of Theorem 2.2, interchanging 0- and 1-edges does not alter the values of $|v(0) - v(1)|$, hence it suffices to consider just types 01 and 00. Nevertheless, the analysis for types 10 and 11 paths will be useful in the study of generalized theta graph.

The EBI sets of P_2, P_3, and P_4 can be easily determined. For $n \geq 5$, all four types of labelings are possible, and $e(0) - e(1)$ could be both ±1 when n is even. Hence we have obtained the following result.

Theorem 3.1 *For $n \geq 5$,*

$$EBI(P_n) = \begin{cases} \{0,1\} & \textit{if } n \textit{ is odd,} \\ \{0,1,2\} & \textit{if } n \textit{ is even.} \end{cases}$$

For small n, $EBI(P_2) = \{2\}$, $EBI(P_3) = \{0\}$, and $EBI(P_4) = \{1,2\}$.

For an n-cycle, since $n \geq 3$, both $e(0)$ and $e(1)$ are at least 1. In each block of consecutive 0- and 1-edges, we find $v(0) = c_i - 1$ and $v(1) = d_i - 1$. Hence $v(0) = e(0) - b$, $v(1) = e(1) - b$, and $v(0) - v(1) = e(0) - e(1)$.

Theorem 3.2 *For $n \geq 3$,*

$$EBI(C_n) = \begin{cases} \{0\} & \textit{if } n \textit{ is even,} \\ \{1\} & \textit{if } n \textit{ is odd.} \end{cases}$$

4 EBI Sets of Generalized Theta Graphs

Let n_{00}, n_{01}, n_{10}, and n_{11} denote the numbers of paths of the respective type in a generalized theta graph. Obviously, $0 \leq n_{00}, n_{01}, n_{10}, n_{11} \leq k$, and $n_{00} + n_{01} + n_{10} + n_{11} = k$. The edge labels on the ith path induces an edge labeling of that path which may not be edge-friendly. Let $e_i(0)$, $e_i(1)$, $v_i(0)$, and $v_i(1)$ represent the value of $e(0)$, $e(1)$, $v(0)$, and $v(1)$, respectively, on the ith path as an individual *stand-alone* path. The value of $v_i(0) - v_i(1)$ depends on the type the ith path belongs to:

$$e_i(0) - e_i(1) = \begin{cases} v_i(0) - v_i(1) & \text{type 01 and 10,} \\ v_i(0) - v_i(1) - 1 & \text{type 00,} \\ v_i(0) - v_i(1) + 1 & \text{type 11.} \end{cases}$$

It follows that

$$e(0) - e(1) = \sum_{i=1}^{n} [e_i(0) - e_i(1)] = n_{11} - n_{00} + \sum_{i=1}^{n} [v_i(0) - v_i(1)].$$

However, $\sum_{i=1}^{n} [v_i(0) - v_i(1)]$ may not equal to $v(0) - v(1)$, because "splicing" the k paths at their endpoints removes the contribution of the vertex labels of the original $2k$ endpoints, and replaces them with the new vertex labels $f^*(u)$ and $f^*(w)$. More specifically,

$$v(0) - v(1) = \sum_{i=1}^{n} [v_i(0) - v_i(1)] - 2n_{00} + 2n_{11} + \delta,$$

where $\delta = \delta(n_{00}, n_{01}, n_{10}, n_{11})$ denotes the contribution of $f^*(u)$ and $f^*(w)$ to $v(0) - v(1)$:

$f*(u)$	–	–	–	0	0	0	1	1	1
$f*(w)$	–	0	1	–	0	1	–	0	1
δ	0	1	−1	1	2	0	−1	0	−2

We deduce that

$$v(0) - v(1) = e(0) - e(1) + n_{11} - n_{00} + \delta.$$

Going over every possible combination of $n_{00}, n_{01}, n_{10}, n_{11}$ that admits an edge-friendly labeling, and for each combination, computing the value of δ, allows us to gather the resulting values of $|v(0) - v(1)|$ to form the EBI set.

If $\ell_i \geq 4$, we can always find an edge-friendly labeling for any $0 \leq n_{00}, n_{01}, n_{10}, n_{11} \leq k$ that satisfy $n_{00} + n_{01} + n_{10} + n_{11} = k$. Here is the reason. We first label the first two and the last two edges of the k paths as follows.

first n_{00} paths	$01\ldots10$
next n_{01} paths	$01\ldots01$
next n_{10} paths	$10\ldots10$
last n_{11} paths	$10\ldots01$

Thus far, we have an equal number of 0- and 1-edges. It is easy to extend this partially completed labeling to an edge-friendly labeling of the entire graph. Furthermore, when $q = \sum_{i=1}^{k} \ell_i$ is odd, $e(0) - e(1)$ could attain the value of both 1 and -1. It remains to determine the value of $n_{11} - n_{00} + \delta$.

For any fixed n_{01} and n_{10}, we have $n_{00} + n_{11} = k - n_{01} - n_{10}$. Hence

$$\Delta \overset{\text{def}}{=} n_{11} - n_{00} + \delta = k - n_{01} - n_{10} - 2n_{00} + \delta.$$

where $0 \leq n_{00} \leq k - n_{01} - n_{10}$.

We first consider $n_{01} + n_{10} \leq 1$. Due to symmetry (we could flip the graph around, or rename u as w, and w as u), we only need to study two cases: $(n_{01}, n_{10}) = (0,0), (1,0)$.

If $n_{01} = n_{10} = 0$, then $\Delta = k - 2n_{00} + \delta$, where $0 \leq n_{00} \leq k$. When $k = 2t$, where $t \geq 2$, we find

	$f^*(u)$	$f^*(w)$	value of Δ	
$0 \leq n_{00} \leq t-1$	1	1	$k - 2 - 2n_{00}$	$k-2, k-4, \ldots, 2, 0$
$n_{00} = t$	–	–	$k - 2n_{00}$	0
$t+1 \leq n_{00} \leq k$	0	0	$k + 2 - 2n_{00}$	$0, -2, \ldots, -(k-2)$

When $k = 2t+1$, where $t \geq 1$, we find

	$f^*(u)$	$f^*(w)$	value of Δ	
$0 \leq n_{00} \leq t$	1	1	$k - 2 - 2n_{00}$	$k-2, k-4, \ldots, 1, -1$
$t+1 \leq n_{00} \leq k$	0	0	$k + 2 - 2n_{00}$	$1, -1, \ldots, -(k-2)$

In other words, all the values between $-(k-2)$ and $k-2$, inclusive, that are congruent to $k \pmod 2$ are attainable.

If $n_{01} = 1$, and $n_{10} = 0$, then $\Delta = k-1-2n_{00}+\delta$, where $0 \le n_{00} \le k-1$. When $k = 2t$, where $t \ge 2$, we have

	$f^*(u)$	$f^*(w)$	value of Δ	
$0 \le n_{00} \le t-2$	1	1	$k-3-2n_{00}$	$k-3, k-5, \ldots, 3, 1$
$n_{00} = t-1$	–	1	$k-2-2n_{00}$	0
$n_{00} = t$	0	–	$k-2n_{00}$	0
$t+1 \le n_{00} \le k-1$	0	0	$k+1-2n_{00}$	$-1, -3, \ldots,, -(k-3)$

When $k = 2t+1$, where $t \ge 1$, we have

	$f^*(u)$	$f^*(w)$	value of Δ	
$0 \le n_{00} \le t-1$	1	1	$k-3-2n_{00}$	$k-3, k-5, \ldots, 2, 0$
$n_{00} = t$	0	1	$k-1-2n_{00}$	0
$t+1 \le n_{00} \le k-1$	0	0	$k+1-2n_{00}$	$0, -2, \ldots, -(k-3)$

Combining these values, we conclude that, for all $k \ge 3$,

$$\Delta = k-2, k-3, \ldots, -(k-3), -(k-2).$$

For $n_{01} + n_{10} \ge 2$, we have

$$n_{11} - n_{00} = k - n_{01} - n_{10} - 2n_{00} \le k - 2 - 2n_{00}.$$

Since δ is at most ± 2, the resulting values of Δ will still be between $k-2$ and $-(k-2)$. In other words, it suffices to consider $n_{01} + n_{10} \le 1$.

Recall that $e(0) - e(1)$ equals 0 if the number of edges $q = \sum_{i=1}^{k} \ell_i$ is even; but equals ± 1 if q is odd, and both ± 1 are attainable in such an event. We have proved the following result about generalized theta graphs.

Theorem 4.1 *For $4 \le \ell_1, \ell_2, \ldots, \ell_k$, where $k \ge 3$, let $q = \sum_{i=1}^{k} \ell_i$, then*

$$EBI(\Theta(\ell_1, \ell_2, \ldots, \ell_k)) = \begin{cases} \{0, 1, 2, \ldots, k-2\} & \text{if } q \text{ is even,} \\ \{0, 1, 2, \ldots, k-1\} & \text{if } q \text{ is odd.} \end{cases}$$

The same argument used above also applies to a more general situation.

Theorem 4.2 *For $2 \le \ell_1, \ell_2, \ldots, \ell_k$, where $k \ge 3$, and $q = \sum_{i=1}^{k} \ell_i \ge 4k$,*

$$EBI(\Theta(\ell_1, \ell_2, \ldots, \ell_k)) = \begin{cases} \{0, 1, 2, \ldots, k-2\} & \text{if } q \text{ is even,} \\ \{0, 1, 2, \ldots, k-1\} & \text{if } q \text{ is odd.} \end{cases}$$

5 The Case of $q < 4k$

When q, the number of edges, is less than $4k$, care must be exercised to ensure that the combinations of n_{00}, n_{01}, n_{10}, and n_{11} are feasible. Furthermore, when q is odd, it may no longer be true that $e(0) - e(1)$ could be both 1 or -1 for any combination of n_{00}, n_{01}, n_{10}, and n_{11}.

Theorem 5.1 *For $2 \leq \ell_1, \ell_2, \ldots, \ell_k$, where $k \geq 3$, and $\sum_{i=1}^{k} \ell_i = 4k - 1$, we have*

$$EBI(\Theta(\ell_1, \ell_2, \ldots, \ell_k)) = \{0, 1, 2, , k - 2\}.$$

Proof. If the generalized theta graph has exactly $4k - 1$ edges, $e(0) - e(1)$ could be 1 or -1. The question is whether both values are attainable. Again, it suffices to consider $(n_{01}, n_{10}) = (0, 0), (1, 0)$.

For $(n_{01}, n_{10}) = (1, 0)$, the proof of Theorem 4.1 reveals that as n_{00} varies from 0 to $k - 1$,

$$\Delta = k - 3, k - 5, \ldots, -(k - 5), -(k - 3).$$

Given any n_{00} and n_{11} that satisfy $n_{00} + n_{11} = k - 1$, we can label $2n_{00}$ interior edges with 1, $2n_{11}$ with 0. The last remaining interior edge can be labeled either 0 or 1, so as to obtain 1 or -1 for $e(0) - e(1)$. Hence

$$v(0) - v(1) = k - 2, k - 3, \ldots, -(k - 3), -(k - 2).$$

For $(n_{01}, n_{10}) = (0, 0)$, we find, as n_{00} varies from 0 to k,

$$\Delta = k - 2, k - 4, \ldots, -(k - 4), -(k - 2).$$

If $1 < n_{00}, n_{11} < k$, both 1 and -1 are possible for $e(0) - e(1)$, as we could label the $2k - 1$ interior edges as follows (this time, $n_{00} + n_{11} = k$):

number of interior edges		
labeled 0	labeled 1	$e(0) - e(1)$
$2n_{11} - 1$	$2n_{00}$	-1
$2n_{11}$	$2n_{00} - 1$	1

If $n_{00} = 0$, we could, at best, label all $2k - 1$ interior edges 0, that only gives us $e(0) - e(1) = -1$, which yields $v(0) - v(1) = k - 3$. Likewise, if $n_{11} = 0$, then $e(0) - e(1)$ could only equal 1, thereby giving $v(0) - v(1) = -(k - 3)$. Therefore, for $(n_{01}, n_{10}) = (0, 0)$,

$$v(0) - v(1) = k - 3, k - 4, \ldots, -(k - 4), -(k - 3).$$

We conclude that the EBI set is $\{0, 1, 2, \ldots, k - 2\}$. ☐

In a similar manner, if $\ell_1 = 1$, the edge uw could be an 0- or 1-edge, which could be regarded as a path of type 00 and type 11 respectively. This forces $n_{00} > 0$, and $n_{11} > 0$ respectively. This in turn implies that the largest absolute value in $v(0) - v(1)$ may not be attainable.

Theorem 5.2 *For* $1 = \ell_1 < \ell_2 \leq \cdots \leq \ell_k$, *where* $k \geq 3$, *and* $q = \sum_{i=1}^{k} \ell_k \geq 4k - 2$,

$$EBI(\Theta(\ell_1, \ell_1, \ldots, \ell_k)) = \begin{cases} \{0, 1, 2, \ldots, k-2\} & \text{if } q \text{ is even,} \\ \{0, 1, 2, \ldots, k-1\} & \text{if } q \text{ is odd,} \end{cases}$$

and if $q = 4k - 3$,

$$EBI(\Theta(\ell_1, \ell_1, \ldots, \ell_k)) = \begin{cases} \{0, 1, 2, \ldots, k-2\} & \text{if } k \geq 4, \\ \{0, 1, 2\} & \text{if } k = 3. \end{cases}$$

Proof. As we have remarked in the proof of Theorem 2.2, we may assume the edge uw is an 0-edge. If $(n_{01}, n_{10}) = (0, 0)$, we have $1 \leq n_{00} \leq k$. As n_{00} varies from 1 to k, we find

$$\Delta = \begin{cases} k-4, k-6, \ldots, -(k-4), -(k-2) & \text{if } k \geq 4, \\ -1, 1, -1 & \text{if } k = 3. \end{cases}$$

If $(n_{01}, n_{10}) = (1, 0)$, we have $1 \leq n_{00} \leq k - 1$. As n_{00} varies from 1 to $k - 1$, we find

$$\Delta = \begin{cases} k-5, k-7, \ldots, -(k-5), -(k-3) & \text{if } k \geq 4, \\ 0, 0 & \text{if } k = 3. \end{cases}$$

Next, we examine the values that $e(0) - e(1)$ could assume.

For $q \geq 4k - 2$, label any $2n_{00} - 1$ interior edges with 1, and any $2n_{11}$ interior edges with 0. Along with the $2n_{00} - 1$ 0-edges and the $2n_{11}$ 1-edges that are incident to u and w, we have assigned an equal number of 0- and 1-edges. It can be easily extended to an edge-friendly labeling of the entire theta graph so that $e(0) - e(1)$ could be either 0, 1, or -1. Therefore the EBI set in such an event is $\{0, 1, 2, \ldots, k-2\}$ if q is even. If q is odd,

$$v(0) - v(1) = \begin{cases} k-3, k-4, \ldots, -(k-2), -(k-1) & \text{if } k \geq 4, \\ 2, 1, 0, -1, -2 & \text{if } k = 3. \end{cases}$$

In either case, the EBI set is $\{0, 1, 2, \ldots, k-1\}$ if q is odd.

The remaining unsettled case is $q = 4k - 3$, in which $e(0) - e(1)$ could only equal to 1 when $n_{00} = k$. This means we could only add 1 to the last entries in the list depicted in the case of $(n_{01}, n_{10}) = (0, 0)$. We find

$$v(0) - v(1) = \begin{cases} k-3, k-4, \ldots, -(k-3), -(k-2) & \text{if } k \geq 4, \\ 2, 1, 0, -1, -2 & \text{if } k = 3. \end{cases}$$

The proof is complete after we take absolute value. □

For $q = 4k - r$, where $r > 1$, the complexity involved in the argument increases as r increases. For $\Theta(2^{[m]}, n)$, where $3 \leq n \leq 2m + 2$, it would be interesting to study how the EBI set changes from the ones described in Theorems 2.1 and 2.2. Likewise, two other families of generalized theta graphs are of particular interest: $\Theta(3^{[m]})$, and $\Theta(3^{[m]}, n)$. We close our discussion with the derivation of $\mathrm{EBI}(\Theta(2^{[m]}, 3))$, and leave the rest as open problems.

Theorem 5.3 *For $m \geq 2$,*

$$EBI(\Theta(2^{[m]}, 3)) = \begin{cases} \{0, 1, 2\} & \text{if } m \text{ is even,} \\ \{0, 1\} & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Edge-friendliness implies $n_{00} - n_{11} = 0, \pm 1$. If $n_{00} = n_{11}$, then the interior edge on P_3 could be labeled either 1 or -1, thus $e(0) - e(1) = \pm 1$. The two end vertices u and w are either both unlabeled, or are labeled differently, hence $\delta = 0$. Consequently, $\Delta = 0$, and $v(0) - v(1) = \pm 1$. Since we can set $n_{00} = n_{11} = 0$, the EBI set always contains 1.

If $n_{00} = n_{11} + 1$, then the interior edge of P_3 must be labeled 1, thus $e(0) - e(1) = 1$. It follows that $v(0) - v(1) = \delta$. In a similar manner, we also find $v(0) - v(1) = \delta$ if $n_{00} = n_{11} - 1$. Since $m \geq 2$, we can set $n_{01} \geq 2$ and $n_{10} = 0$ such that $n_{00} + n_{11} = m + 1 - n_{01} \leq m - 1$ is odd. Note that $f^*(u) = 0$, and $f^*(w) = 1$, hence $v(0) - v(1) = \delta = 0$. Therefore the EBI set always contains 0.

The remaining question is: when could $n_{00} + n_{11}$ be odd (so that $n_{00} + n_{11} = \pm 1$) and $\delta = \pm 2$? Observe that when $|n_{01} - n_{01}| \geq 1$, the two end vertices u and w are either both unlabeled, or are labeled differently. In either case, δ equals 0 or ± 1. If $n_{01} = n_{10}$ and $n_{00} - n_{11} = \pm 1$, then both u and w will be labeled the same, hence $\delta = \pm 2$. This happens when only $m + 1$ is odd. This completes the proof. □

6 Closing Remarks

We have demonstrated that finding the edge-balance index set of a graph could be tedious, if not difficult, in general. Unlike balance index set, which could be obtained using algebraic approaches [4, 10], no such methods seem to be possible for edge-balance index sets. A reason is: f^* does not take into account the difference between the numbers of 0- and 1-edges incident to a vertex. Consequently, we cannot use the information from the degree sequence. This makes it difficult to construct a simple algebraic approach. We invite the readers to join us tackle this problem.

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