ON EDGE-MAGIC CUBIC GRAPHS CONJECTURE

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Abstract
A graph G=(V,E) is said to be edge-magic if there exists a bijection f: E {1,2,...,|E|} *such that the induced mapping* $f^+ : V \to N$ *defined by* $f^+(u) = \{f(u,v) : (u,v) \in E\} \pmod{|V|}$ *is a constant map. In 1993 Lee conjectured that a cubic* (p, q) *graph is edge-magic if and only if* $p \equiv 2 \pmod 4$. *We present positive results for several classes of cubic graphs. A new conjecture with respect to the edge-magic nature of permutation graphs is presented.*

1. Introduction

A graph G on p vertices and q edges is said to be edge-magic if there exists a bijection between the integers (edge weights) 1, 2, ..., q and the edges of G such that for any x and y, vertices of G, the sum of edge weights incident to x (mod p) equals the sum of edge weights incident to y (mod p) [6]. Hartsfield and Ringel[5] have presented results on a special case of edge-magic known as supermagic. The theory of supermagic graphs is a subtheory of magic graphs and is discussed in [1, 2, 3, 4]. Lee conjectures that $p \equiv 2 \pmod 4$ is a necessary and sufficient condition for any cubic graph on p vertices to be edge-magic. Computer searches on several cubic graphs have suggested that this might be the case. A proof, however, will require a great deal of insight into the structure of cubic graphs. We present some positive results for this conjecture.

2. Prism

The simplest cubic graph is the prism, $C_n \times K_2$ where n = 3. Let G be a prism with p = |V| = 6 and q = |E| = 9. The set of edge-magic labelings must come from the set of all permutations on {1,2, . . ., |E|}. Thus, if an exhaustive search is undertaken, there are q! possible labelings to test. For q = 9 this is not a difficult task and, in fact, such a computer search yields 336 labelings of G that are edge-magic. The goal of this search is not to find all such labelings, but to find labelings whose pattern will generalize to larger members of the same family of cubic graphs. Searching through 336 labelings for patterns that generalize is not trivial. Fortunately, these are not all unique mappings.

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The first method of eliminating equivalent mappings is in recognizing modulo equivalence. For example, the edge-magic labelings (1, 2, 3, 9, 8, 7, 4, 5, 6) and (1, 2, 9, 3, 8, 7, 4, 5, 6) are equivalent since 9 (mod p) = 3 (mod p). Further reductions due to rotations and reflections can be made based on the symmetry of G. Reducing the set of edge-magic solutions using these techniques results in 4 distinct mappings (Figure 1).

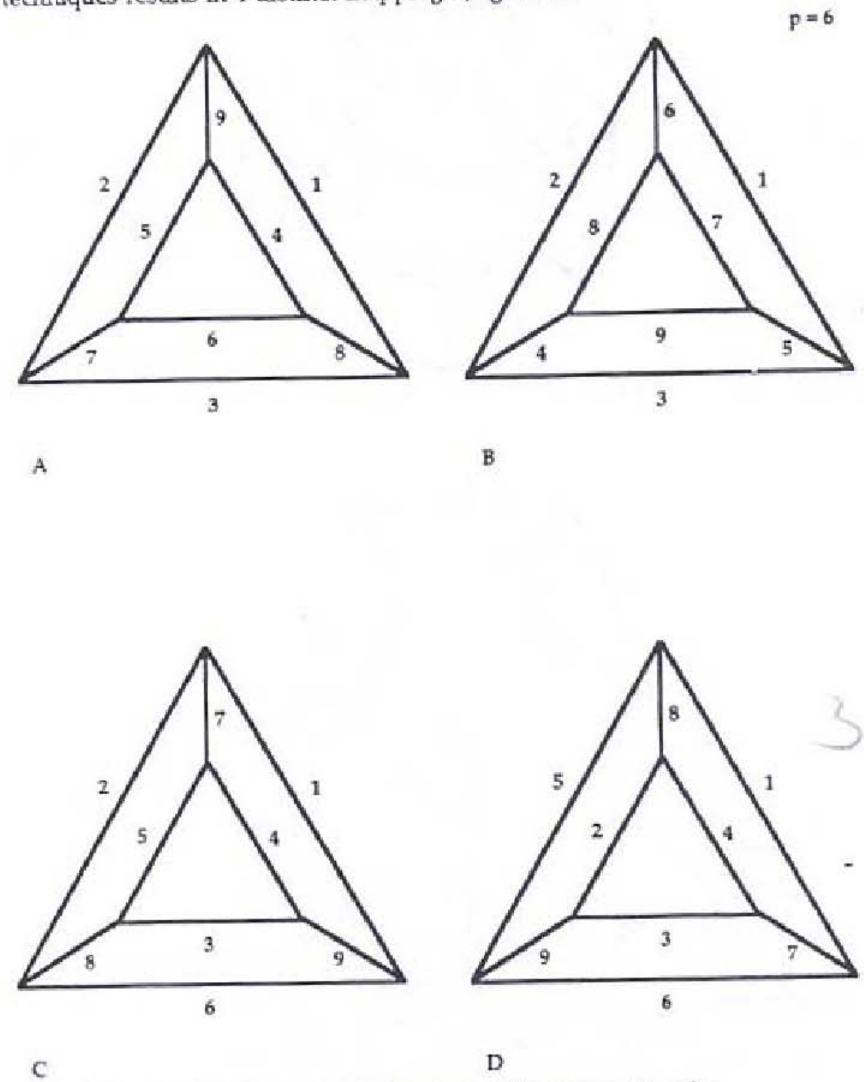

Figure 1. The four unique solutions (after accounting for isomorphisms and modulo equivalence) on $C_3 \times K_2$.

The labelings of Figures 1A and 1B are generalized, using Algorithms 1A a
1B, to any cubic prism that satisfies p ≡ 2 (mod 4).

Algorithm 1A: To produce an edge-magic labeling for any cubic prism $C_n \times K$ where $p = 2n \equiv 2 \pmod 4$ and $q = |E| = 3p/2$:

1) Starting at any edge on one of the cycles: In a clockwise direction, label every other edge on this cycle with successive integers 1, 2, ..., p/2.
2) Upon arriving at previously labeled edge E[1], continue the labeling in the same direction and manner beginning with the edge of the other cycle that is parallel to E[1].
3) Upon labeling edge E[p], label the edge that joins the vertex common to E[p] and $E[p - \lfloor n/2 \rfloor]$ to the other cycle as p+1.
4) In a counter-clockwise direction, successively label the remaining edges as p+2, p+3, ..., q.

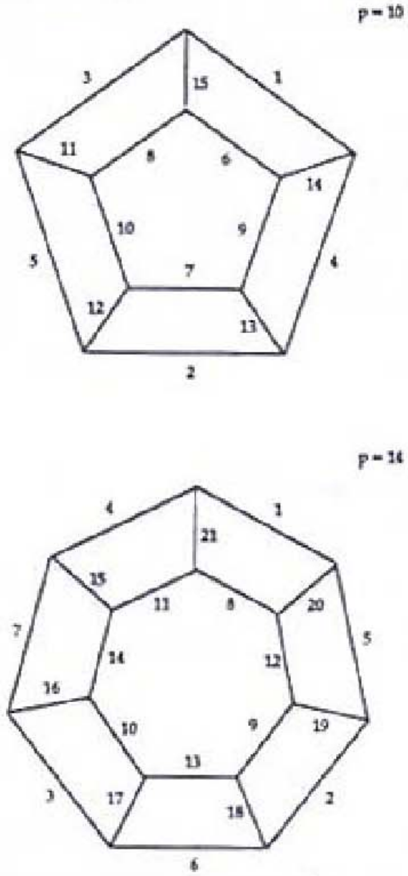

Figure 2. Application of Algorithm 1A to $C_5 \times K_2$ and $C_7 \times K_2$.

Algorithm 1B: To produce an edge-magic labeling for any cubic prism $C_n \times K$ where $p = 2n \equiv 2 \pmod 4$ and $q = |E| = 3p/2$:

1) Starting at any edge on one of the cycles:
   In a clockwise direction, label every other edge on this cycle with successive integers 1, 2, ..., p/2.
2) Upon labeling edge $E[p/2]$, label the edge that joins the vertex common to $E[p/2]$ and $E[(p/2) - \lfloor n/2 \rfloor]$ to the other cycle as $(p/2) + 1$.
3) In a counter-clockwise direction, successively label connecting edges $(p/2) + 2$, $(p/2) + 3$, ..., p.
4) Label remaining edges E_j, ($j = p+1, p+2, \ldots, q$), with $p + i$ where i is the label of the edge parallel to E_j.

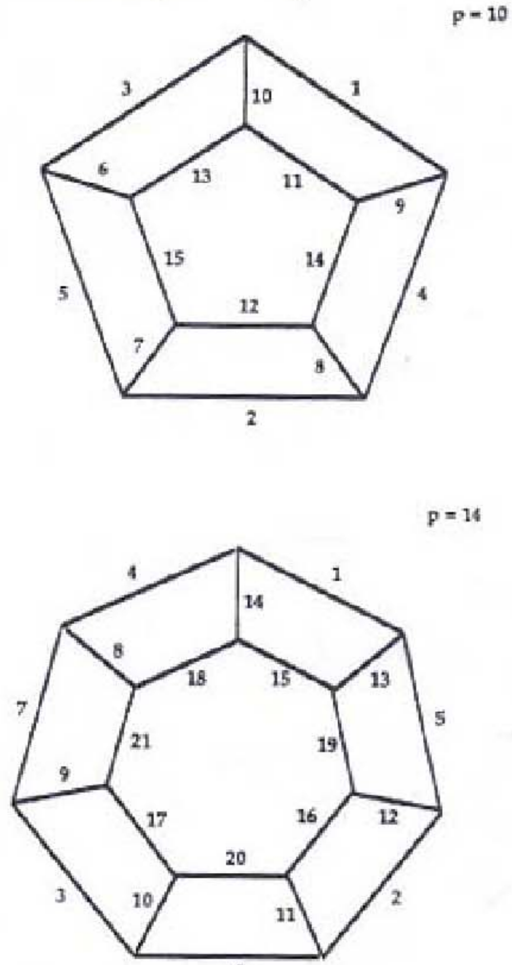

Figure 3. Application of Algorithm 1B to $C_5 \times K_2$ and $C_7 \times K_2$.

Algorithms 1A and 1B take advantage of the property that a cubic graph on p vertices will have 3p/2 edges and the set of integers from 1 to 3p/2 can be partitioned into the sets {1, 2, ..., p/2}, {(p/2)+1, (p/2)+2, ..., p}, and {p+1, p+2, ..., 3p/2}. The development of Algorithm 1B was simplified since the first and third sets are equivalent (mod p). Consider the prism to be two concentric cycles as in Figure 4. If edge labelings can be found using two of the three sets of this partition, such that the sum of adjacent edges produce a sequential and consecutive labeling on the vertices of each cycle, and if the labelings on both cycles are equivalent (mod p), then labeling the connecting edges from the unused set of consecutive integers follows. The modulo equivalence of the first and third sets enables a complete solution of the prism from the sequential and consecutive labeling on the vertices of one cycle.

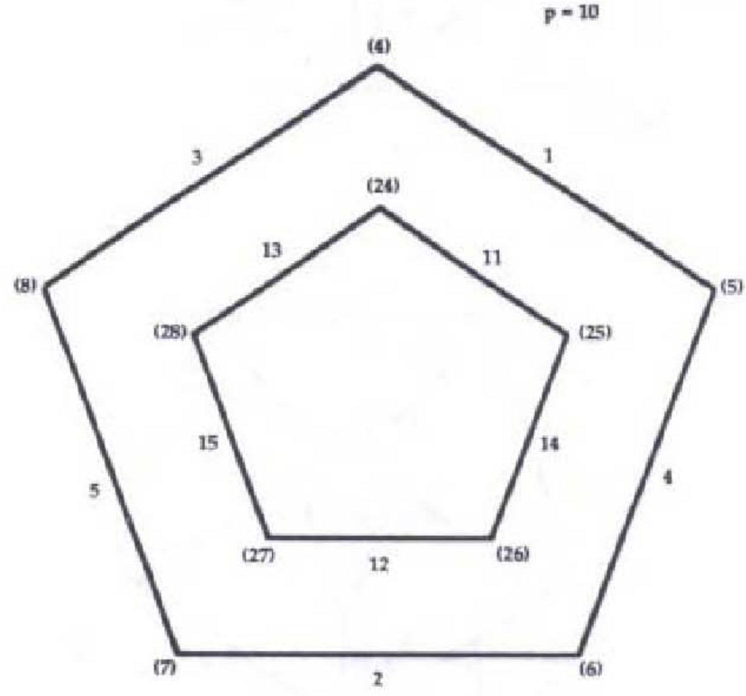

Figure 4. The concentric cycles of $C_5 \times K_2$ with the induced vertex labeling.

<u>Theorem 1</u>: *A prism $C_n \times K_2$ is edge-magic if and only if n is odd.*

3. <u>Moebius Ladder</u>

The Moebius ladder (Figure 5) is related to the prism. The diminished symmetry, however, makes an edge-magic analysis more difficult. An exhaustive search on the Moebius ladder M(6, 9) produces 1440 edge-magic labelings. Reductions in this number due to modulo equivalence leaves too many labelings to search for patterns. Algorithm 1B allows us to treat parallel sides as interchangeable. Since labels on these pairs differ by exactly p, they are modulo equivalent. For the Moebius ladder, this means that the crossing edges are modulo equivalent. A direct application of Algorithm 1B produces an edge-magic labeling on any Moebius ladder M(p, q) where $p \equiv 2 \pmod 4$.

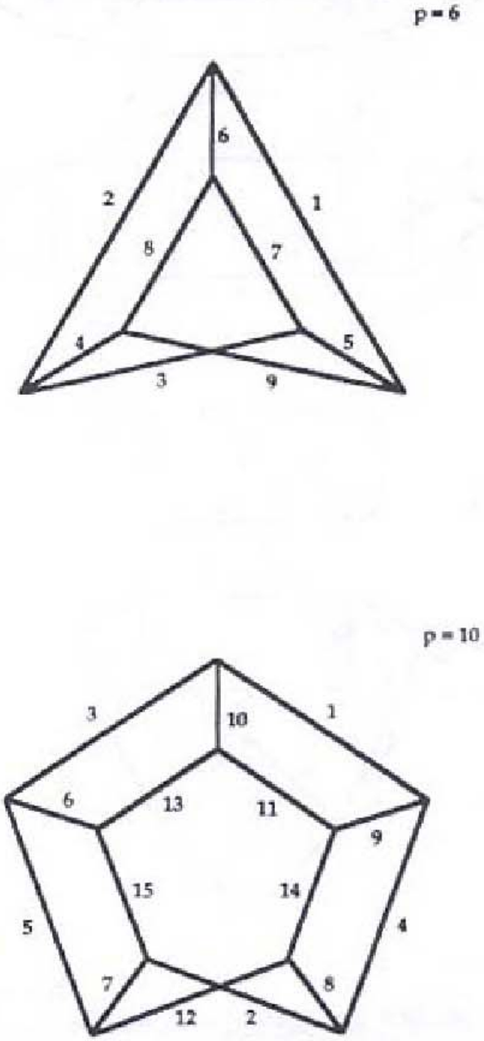

Figure 5. Application of Algorithm 1B to M(6, 9) and M(10, 15).

<u>Theorem 2</u>: *The Moebius ladder M(p, q) is edge-magic if and only if* $p \equiv 2 \pmod 4$.

4. Petersen Graph

The analysis of the prism does not apply to the Petersen graph (Figure 6). Relationships between vertices are more interdependent than in the prism or Moebius ladder. Figure 6 shows two edge-magic labelings of the Petersen graph T(10, 15). A general solution is not apparent in these labelings and a computer search on the 15! permutations is planned.

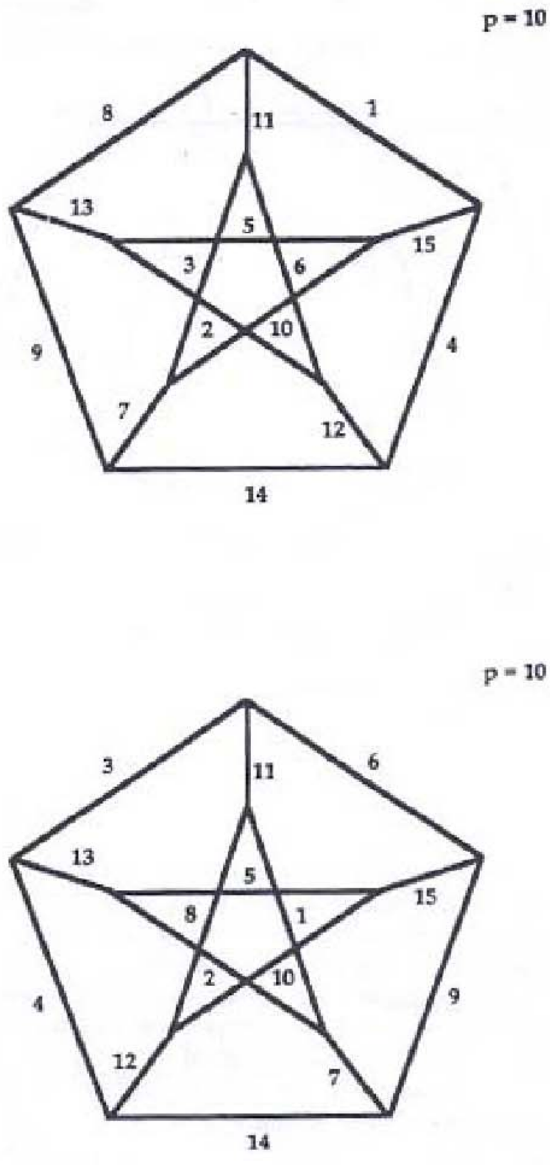

Figure 6. Two edge-magic solutions to T(10, 15).

5. Other results

Edge-magic labelings for some other classes of cubic graphs are presented in Figure 7. There is more work to be done toward discovering the relationships between cubic structures and Lee's conjecture. Attempting to find general solutions to produce labelings is only one approach to understanding the nature of edge-magic.

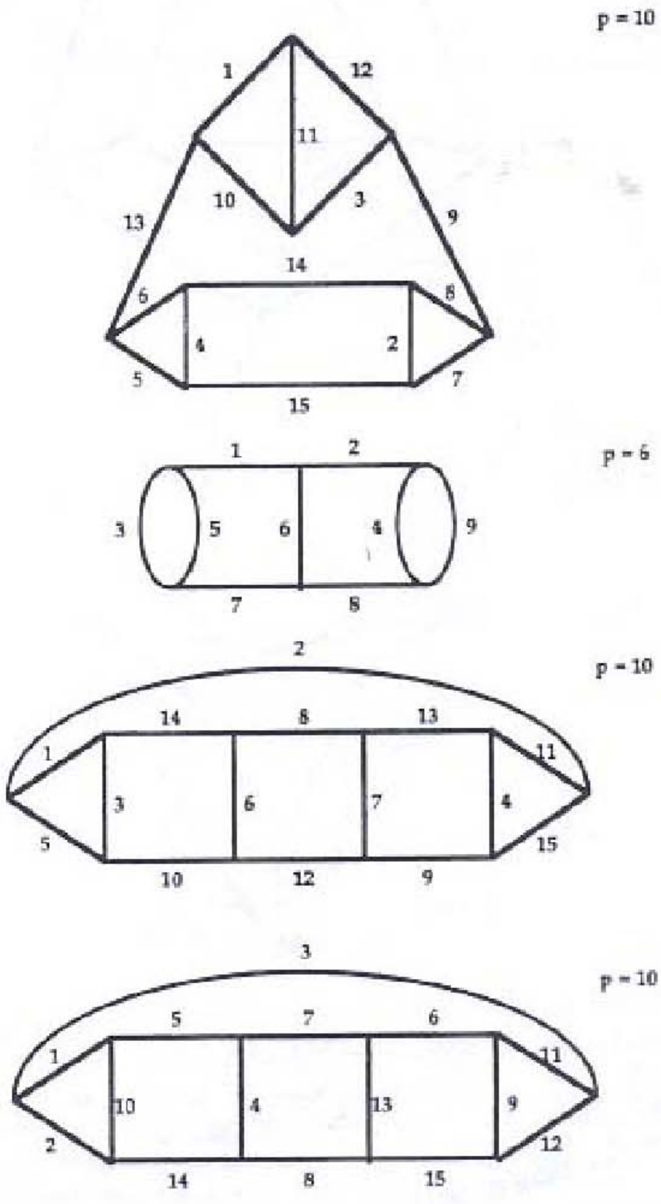

Figure 7. Some edge-magic solutions for other cubic graphs.

Finally, we consider a special case of the cubic graph conjecture. Given a (p, q) graph G, and a permutation $f \in S(p)$, we define the permutation graph $P(G, f)$ as the graph consisting of two copies of G with $V(P(G, f)) = \{v_1, \ldots, v_p, v_1^*, \ldots, v_p^*\}$. $E(P(G, f)) = E(G) \cup E(G^*) \cup \{(v_i, v_{f(i)}^*) : i = 1, \ldots, p\}$. We invite the reader to consider this special case of Lee's conjecture:

Cubic Permutation Graph Conjecture: For a cycle C_n, and any permutation $f \in S(n)$, the permutation graph $P(C_n, f)$ is edge-magic if and only if n is odd.

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