On The Edge-magic Cubic Graphs and Multigraphs

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ABSTRACT

If G is a (p,q) graph in which the edges are labeled 1,2,3,...q so that the vertex sums are constant, mod p, then G is said to be edge-magic. It is conjectured by Lee [9] that any simple cubic graph with p≡2(mod 4) vertices is edge-magic. Lee and Shiu [20] showed that the conjecture is not true for disconnected cubic graphs and multigraphs. We exhibit here three counterexamples of smallest order of this conjecture. Two of them are disconnected simple graphs and one is a connected simple graph.

1. Introduction. All graphs in this paper are simple graphs with no loops or multiple edges. Graceful labelings were first introduced by Alex Rosa as means of attacking the problem of cyclically decomposing the complete graph into other graphs. Since Rosa's original article, literally hundreds of papers have been written on graph labelings [1]. Another dual concept of graceful labeling on graphs which was called edge-graceful was introduced by S.P. Lo [12] in 1985.

A graph G= (V, E) with p vertices and q edges is said to be **edge-graceful** if there is a bijection f : E->{1, 2,..., q} such that the induced mapping f^+:V--> $\mathbf{Z}_p$, given by $f^+(u) = \Sigma\{f(u,v)$: (u,v) in E} (mod p) is a bijection. Figure 1 shows a graph with 8 vertices and 12 edges which is edge-graceful.

A necessary condition of edge-gracefulness is (Lo [12])

$$q(q+1) \equiv \frac{p(p-1)}{2} \pmod{p} \qquad (1)$$

This latter condition may be more practically stated as q(q+1) ≡0 or p/2 (mod p) depending on whether p is odd or even. (2)

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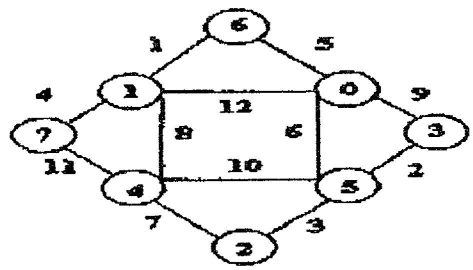


Figure 1.

Lee, Lee, Murthy [6] showed that if G is a (p,q)-graph with p≡ 2 (mod 4) then G is not edge graceful. (3)

Lee proposed the following tantalizing conjecture

Conjecture (Lee [8]): The Lo condition (2) is sufficient for a connected graph to be edge-graceful.

A sub-conjecture of this has also not yet been proved:

Conjecture (Lee [7]): All odd-order trees are edge-graceful.

In order to work on these conjectures, another dual concept of edge-graceful graphs was introduced in 1992. Let G be a (p,q) graph in which the edges are labeled 1,2,3,...q so that the vertex sums are constant, mod p. Then G is said to be **edge-magic.**. The concept of edge-magic graphs was introduced by the first author, Seah and Tan [12]. A necessary condition for a (p,q)-graph to be edge-magic is q(q+1)≡ 0 (mod p). However, this condition is not sufficient. There are infinitely many connected graphs such as trees, cycles satisfying this condition that are not edge-magic. A complete graph K_n is edge-magic if and only if n≠ 0,3 (mod4). Figure 2 shows that K_5 is edge-magic.

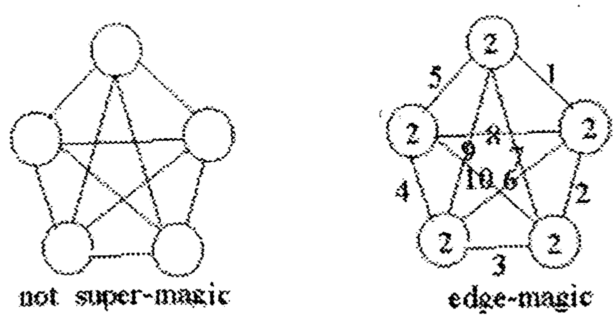


Figure 2

Stewart [20,21] defined a graph **supermagic** if the edges are labeled 1,2,3,...q so that the vertex sums are constant. He completely described the supermagic complete graph in [21]. For a generalization of this result see [5]. Hartsfield and Ringel in [3] exhibit some new supermagic graphs. A general construction of supermagic graphs is considered in [19].

The cartesian product of two paths is frequently called the **grid** graph . The cartesian product of two cycles is called the **torus** graph. It was shown in [16] that the torus graph C_m x C_n is edge-magic for all m,n >2.

Lee, Pigg and Cox [9] showed that $C_n \times K_2$ is edge-magic if and only if n is odd and ≥ 3.

Karl Schaffer and Sin-Min Lee [16] have shown that if G and H are both odd-order, regular, edge-graceful graphs, where G is d-regular with m vertices, H is k-regular with n vertices, and GCD(d,n) = GCD (k,m) =1, then G x H is edge-graceful. In particular, they showed that the torus graph $C_{2i+1} \times C_{2j+1}$ is edge-graceful.

In 1993 Lee conjectured that every connected simple cubic graph G with $p \equiv 2 \pmod 4$ is edge-magic. Lee, Pigg and Cox [9] showed that the conjecture is true for prisms and other cubic graphs.

Since cycles are not edge-magic, it is natural to investigate cubic graphs which are edge-magic. All cubic graphs of order $p \equiv 0 \pmod 4$ are not edge-magic, In 1993 Lee conjecture that every connected simple cubic graph G with $p \equiv 2 \pmod 4$ is edge-magic. It is unlikely that a general proof can be given. Lee, Pigg and Cox [9] showed that the conjecture is true for prisms and other cubic graphs.

The conjecture is not true for cubic multigraphs [19]. The simplest example is given by the following cubic multigraph of order 6 (Figure 3).

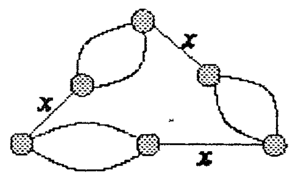


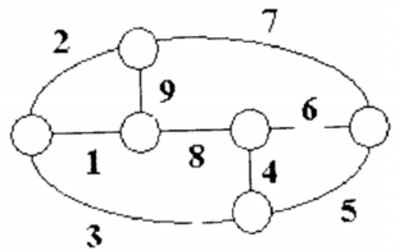
Figure 3.

In section 2 and 6 we show that the conjecture of Lee is not true for disconnected simple graph and connected cubic simple graphs. Some conjectures are proposed in the last section.

The problem of finding an edge-magic labeling of a graph satisfying the necessary condition is, in general NP-complete. Several classes of graphs had been shown to be edge-magic ([12,13,16,17,18,19]). For more conjectures and open problems on edge-magic graphs the reader is referred to [12,13,20]. The reader should also see the survey article of Gallian [1] for various labeling problems.

2. Disconnected Edge-magic cubic multi-graphs

Finding an edge-magic labeling of a graph is related to solving a system of linear Diophantine equations. In general it is difficult to find an edge-magic labeling of a graph. For example the following cubic graph with 9 edges in Figure 4 is edge-magic. Among all the 362889 possible edge labelings only 672 are edge-magic .



sum=0

Figure 4

The concept of edge-magic graphs can be generalized in the following way:

Suppose a (p,q)-graph G is given. If $k \geq 1$ and $r \geq 2$ are given and there exists a bijection $f: E \rightarrow \{k, k+1, \ldots, k+q-1\}$ with the property that the induced mapping $f+: V \rightarrow Zr$ $f+(u) = \Sigma \{ f(u,v) : (u,v) \varepsilon E\}$ (mod r) is a constant, we say that G is (k,r)-edge-magic. For convention, we denote $Z_0=Z$. The usual edge-magic is (1,p)-edge magic in this sense and the super edge-magic is (1,0)-edge-magic.

In this paper, we will denote the disjoin union of two graphs G and H by G+H. Recently, Shiu and Lee [19] showed that the ladder graphs L_n, where L_n is $P_n x K_2$ by adding one more parallel edge on the two ends of the ladder, is edge-magic when n is even. It is also proved that for all odd $n>1$, $n(K_4+K_2[3])$ is edge-magic.

However, not every disconnected cubic simple graphs satisfy the necessary condition of edge-magic graphs are edge-magic. Consider the following disconnected cubic simple graph of order 10 (Figure 5). It is not edge-magic.

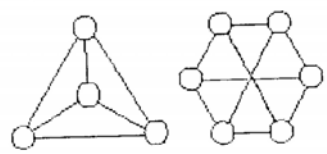


Figure 5.

The following result is established in [19].

Theorem 1.If G is (k,r)-edge-magic then G is (k,s) -edge-magic for any divisor s of r.

Using the above result we can show that

Theorem 2. The simple graph $K_4+ K_{3,3}$ is not edge-magic.
Proof. Assume $K_4+ K_{3,3}$ is edg-magic. Then by Theorem 1 it is (1,5)-edge-magic.
We want to show that it is impossible for $K_4+ K_{3,3}$ to be (1,5)-edge-magic.

Suppose $K_4 + K_{3,3}$ has a (1,5)-edge-magic. Suppose the component K_4 has labeling as Figure 6. Then from $(a+c-d)+(b+c-d)=a+b$ we conclude that $c=d$. Thus K_4 must have the labeling as Figure 6.b.

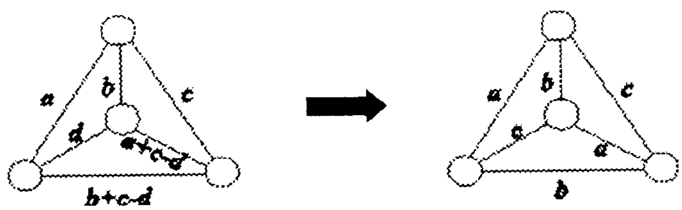


Figure 6 a **Figure 6.b**

Let the other two numbers different from a,b,c be x and y. Then $\{a,b,c,x,y\}=\{0,1,2,3,4\} = Z_5$

We have to label the edges of $K_{3,3}$ by $\{a,b,c,x,x,y,y\}$. The only possible case are:

Case 1. $\{a,b,c\}= \{0,1,4\}$ $\{x,y\}=\{2,3\}$

Case 2. $\{a,b,c\}= \{0,2,3\}$ $\{x,y\}=\{1,4\}$

However, all these cases show that it is impossible for $K_{3,3}$ has a (1,5)-edge-magic labeling. Therefore $K_4 + K_{3,3}$ is not edge-magic.

Using the similar technique as the proof of Theorem 2, we can show that another disconnected cubic simple graph of order 10 (Figure 7) also possesses the same property.

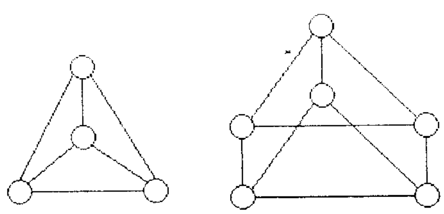


Figure 7.

Theorem 3. The cubic simple graph $K_4 + K_2 \times C_3$ is not edge-magic.

In [19], Siu and the first author showed that $n(4 K_4 + K_{3,3})$ is edge-magic f
all $n \geq 1$.

It was shown in [20] that $K_4 + nK_2[3]$ is edge-magic for n=1 and 3. (Figure 8

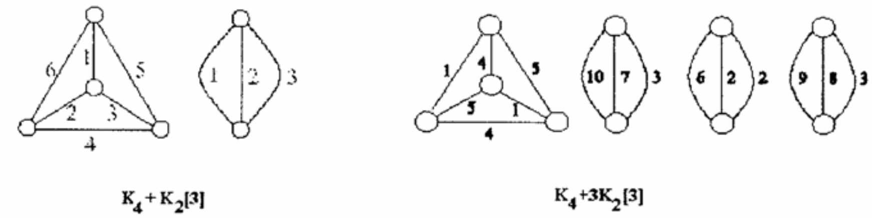


Figure 8.

Using the systems of Diophantine equations associated with the graphs we can show that

Theorem 4 The graphs $K_4+5K_2[3]$ and $K_4+7K_2[3]$ are edge-magic.
Proof. See the following labeling for $K_4+5K_2[3]$. (Figure 9).

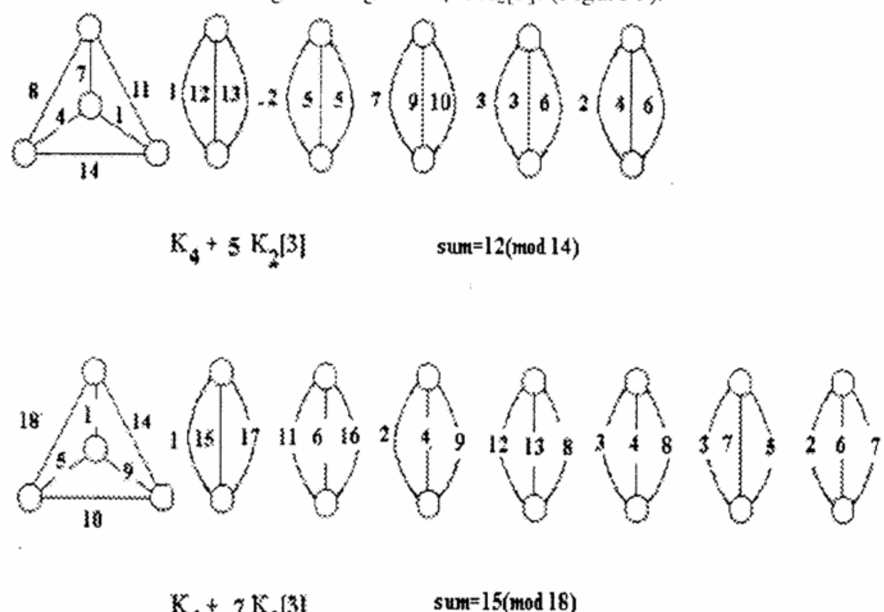


Figure 9

The graph K_4 +5K_2[3] has p=14 and q=21. We see that the labeling
f_E→{1_2_..._14}∪{1_2_..._7}with f⁺_u_=12_mod 14_.
The graph K_4 +7K_2[3] has p=18 and q=27. We see that the labeling
f_E→{1_2_..._18}∪{1_2_..._9}with f⁺_u_=15_mod 18_ .

We shall describe a useful technique to construct new cubic graphs from old ones. For any cubic graph G, we can construct new cubic graphs by an insertion method. Here we add new vertices in G, each of which is the "mid-point" of an existing edge, and join them in pairs to get a cubic graph.

Given a cubic graph G and two edges e_1, e_2 of G, subdivide e_1 and e_2 with two new vertices u,v respectively, and then join u,v. The resulting graph is a cubic graph. We will denote this new cubic graph by $G\Delta 1\{e_1,e_2\}$.

The above construction can be generalized to $k\geq 2$ as follows: Given a cubic graph G and two edges e_1, e_2 of G, subdivide e_1 and e_2 with k new vertices $\{u_1,u_2,\ldots,u_k\}$ $\{v_1,v_2,..,v_k\}$ respectively, and then join u_i,v_i for $i=1,\ldots,k$. The resulting graph is a cubic graph. We will denote this new cubic graph by $G\Delta k\{e_1,e_2\}$.

We introduce here a cubic multigraph X of order 6 which is edge-magic (Figure 10).

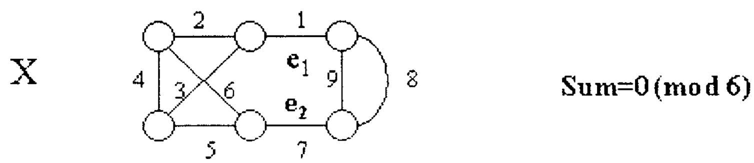


Figure 10.

Theorem 5. The cubic multigraph $X\Delta 2\{e_1,e_2\}$ is edge-magic.
Proof. An edge-magic labeling is given as follows (Figure 11):

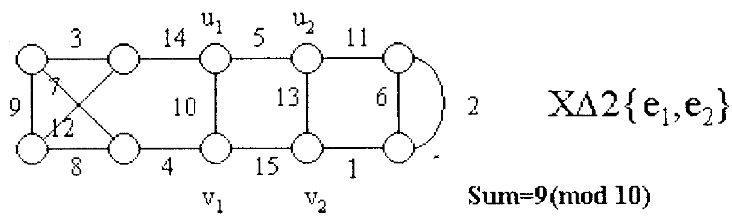


Figure 11.

3. Edge-magic simple cubic graphs obtained from prisms.

Theorem 6. For any $n\geq 2$, and any two parallel edges e_1,e_2 of $C_{2n}\times K_2$, the cubic graph $(C_{2n}\times K_2)\ \Delta 1\{e_1,e_2\}$ is edge-magic.
Proof. If e_1,e_2 are parallel edges then $(C_{2n}\times K_2)\ \Delta 2\{e_1,e_2\}$ is isomorphic to $C_{2n+1}\times K_2$ which is edge-magic by Theorem of Lee et al [9].

The above statement is also true for some examples where e_1,e_2 are not parallel.

Example 1. We illustrate here two examples for the cubic graph $C_4\times K_2$. (Figure 12)

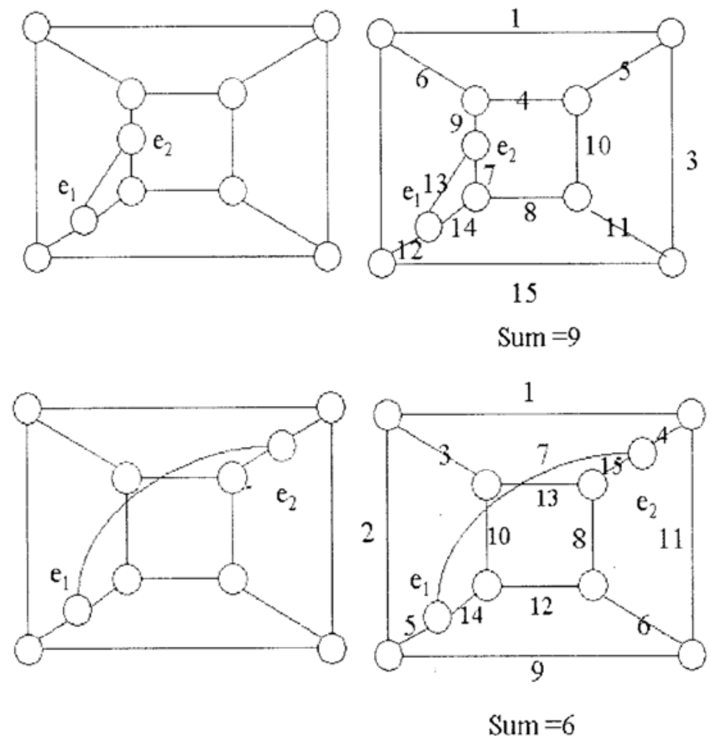


Figure 12.

Example 2. The cubic graph $C_3 \times K_2$ is edge-magic. We can construct a new cubic graph $(C_3 \times K_2)\,\Delta 2\{e_1, e_2\}$ of order 10 as follows.(Figure 13)

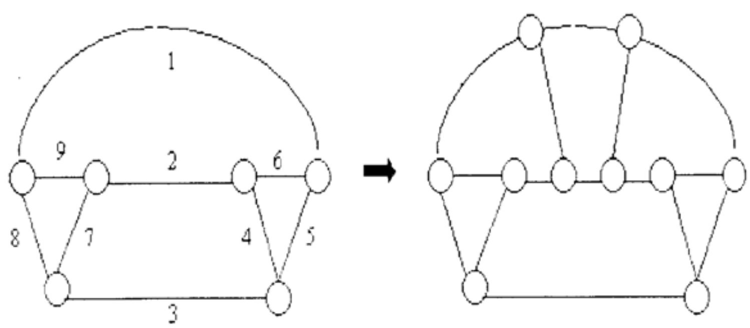


Figure 13

Figure 14 shows the cubic graph $(C_3xK_2)\ \Delta 2\{e_1,e_2\}$ with four different edge-magic labelings, with sums 0, 2,6 and 7.

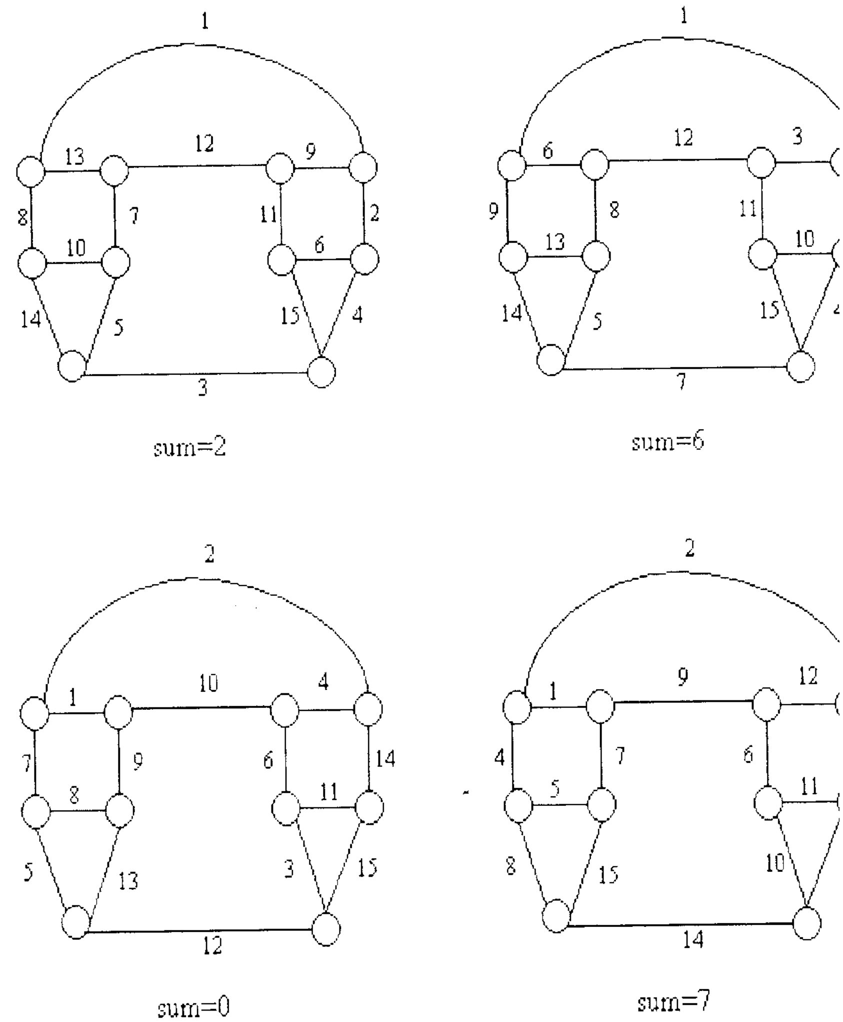


Figure 14.

4. Edge-magic simple cubic graphs obtained from $K_2[3]$

Using the above insertion method we can add new vertices and edges to $K_2[3]$ to form a series of cubic graphs.

The following examples show that this method will provide some ne cubic edge-magic graphs from the old one.

Example 3

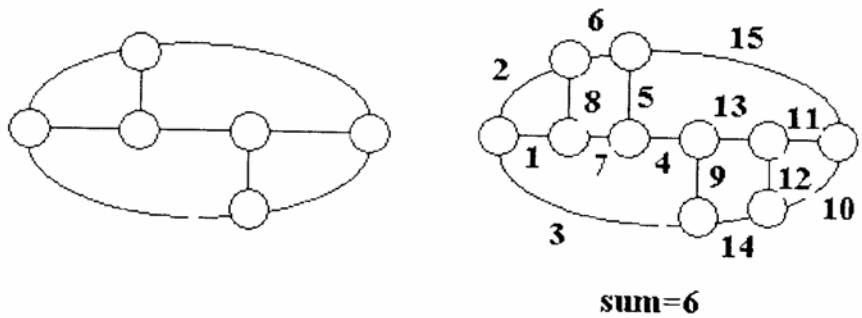


Figure 15.

Example 4.

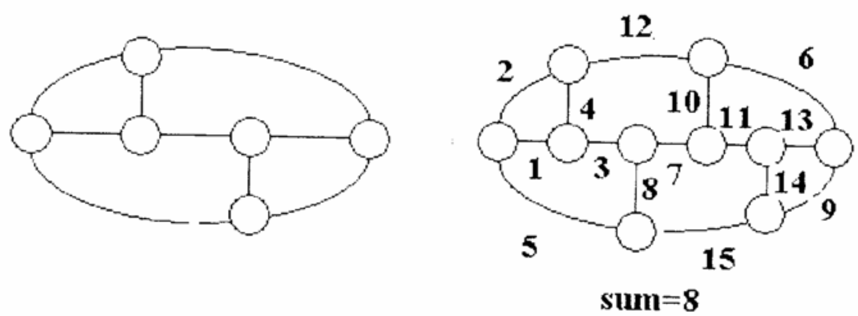


Figure 16.

5. Edge-magic simple cubic graphs obtained from Mobius ladders

Richard Guy and Harary [2] called the following cubic graphs Mobius ladders.

For $n \geq 3$, we denote by $M(n)$ the cubic graph with $V(M(n))=\{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n\}$ and $E(M(n))=\{(u_i,u_{i+1}): 1<i<n-1\}\cup\{(u_n,v_1),(u_1,v_n)\}\cup\{(v_i,v_{i+1}): 1<i<n-1\}\cup\{(u_i,v_i): 1<i<n-1\}$.

Theorem 7. The cubic graph $M(n)$ is edge-magic for all odd $n\geq 3$.

Proof. The graph $M(n)$ has $2n$ vertices and $3n$ edges. Since the addition of the edge labels are modulo $p=2n$, we will labeled the edges by $\{1,2,\ldots,n,n+1,\ldots,2n\} \cup \{1,2,\ldots,n\}$ instead of $\{1,2,\ldots,3n\}$.

We label the edges $\{(u_i,u_{i+1}): 1<i<n-1\}\cup\{(u_n,v_1),(u_1,v_n)\}\cup\{(v_i,v_{i+1}): 1<i<n-1\}$ by the numbers $\{1,2,\ldots,2n\}$ in the following way:

Assume n=2k+1. Along the cycle of M(n) clockwise we define the edge labeling

$f((u_i,u_{i+1})) = (i+1)/2$ for all odd $i \neq n$

$f(u_n,v_1) = (n+1)/2,$

$f(v_{2j},v_{2j+1}) = (n+1)/2 + j$ for j=1,2,...,k.

Thus we will fill the edges with one jump along the cycle with numbers {1,2,...,n}. For the remaining unlabeled edges, we start with the opposite edge of (u_1,u_2), namely (v_1,v_2) and fill it clockwise with n+1, n+2,...,2n.

With this labeling, we see that now u_1 has sum 3k+3,u_2 has sum 3k+4,..., and u_n has sum 3k+ 2+n.

The vertex v_1 has sum 3k+3,v_2 has sum 3k+4,..., and v_n has sum 3k+ 2+n.

Now we will label the edges $\{(u_i,v_i): i=1,2,...,n\}$ by {1,2,...,n} in the reverse order. Then this is an edge-magic labeling of M(n).

Example 5. Figure 17 shows that M(3) and M(5) are edge-magic under the labeling scheme of the above theorem..

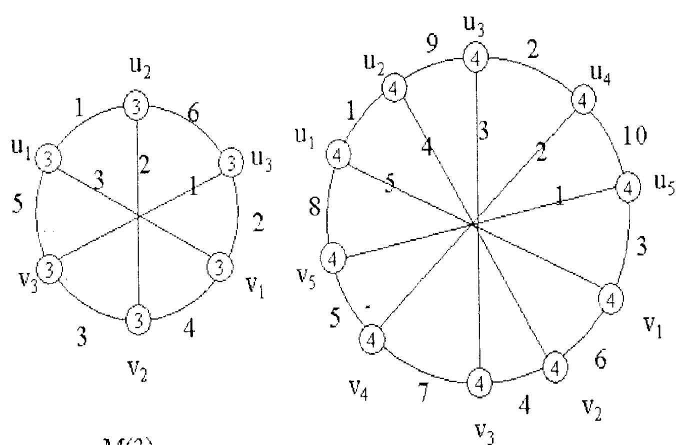


Figure 17.

Using the insertion method we can add new vertices and edges to M(2k+1) to form a series of cubic graphs. We can see that some of them are edge-magic.

Example 6. We add two parallel edges in M(3) and obtain a new edge-magic cubic graph.

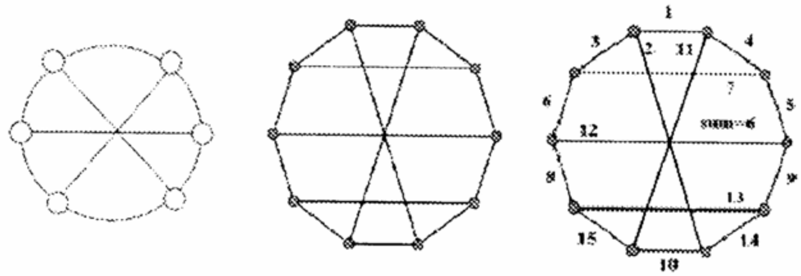
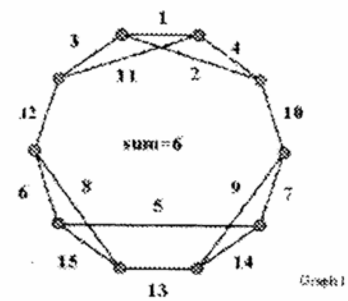
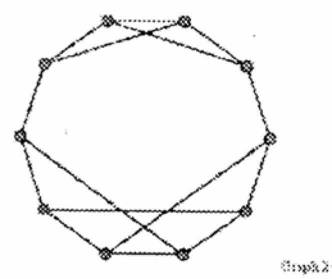


Figure 18.

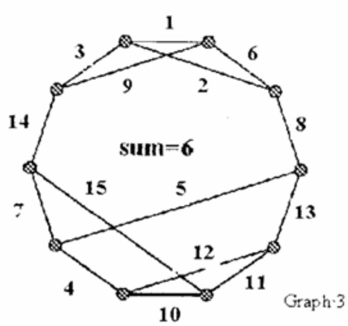
6. Edge-magic Hamiltonian cubic graphs of order 10.

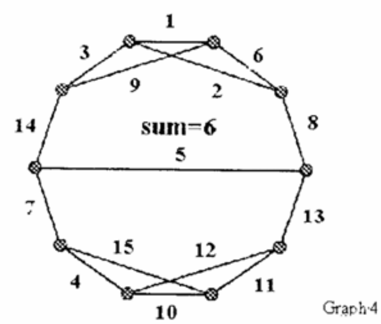
Pigg, Cox and the first author [9] showed that the Petersen graph is edge-magic. Among the 19 non-isomorphic connected cubic graphs of order 10 there are 17 Hamiltonian graphs. We show in Figure 19 that only 16 of them are edge-magic.

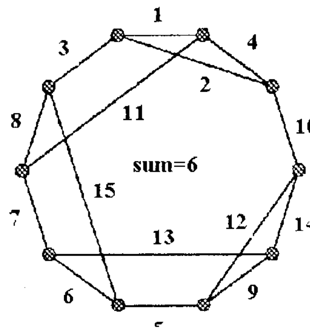




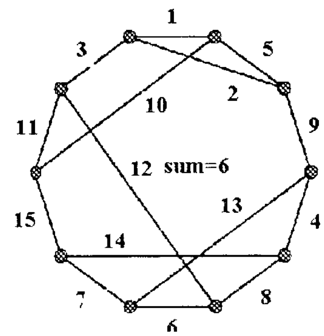
It is not edge-magic



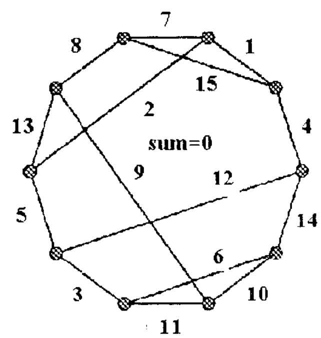




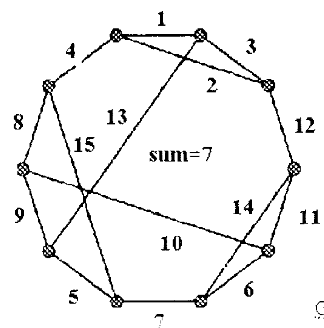
Graph 5



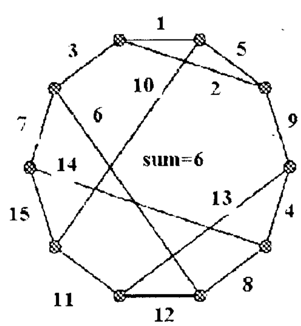
Graph 6



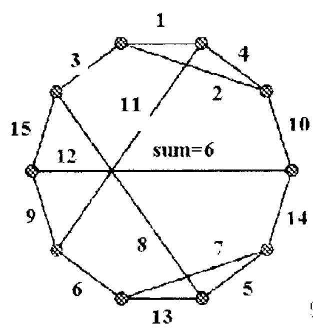
Graph 7



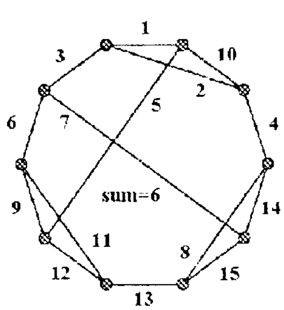
Graph 8



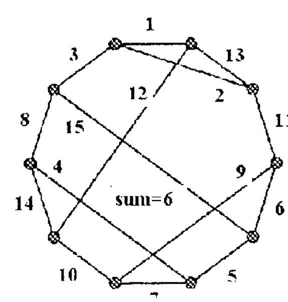
Graph 9



Graph 10



Graph 11



Graph 12

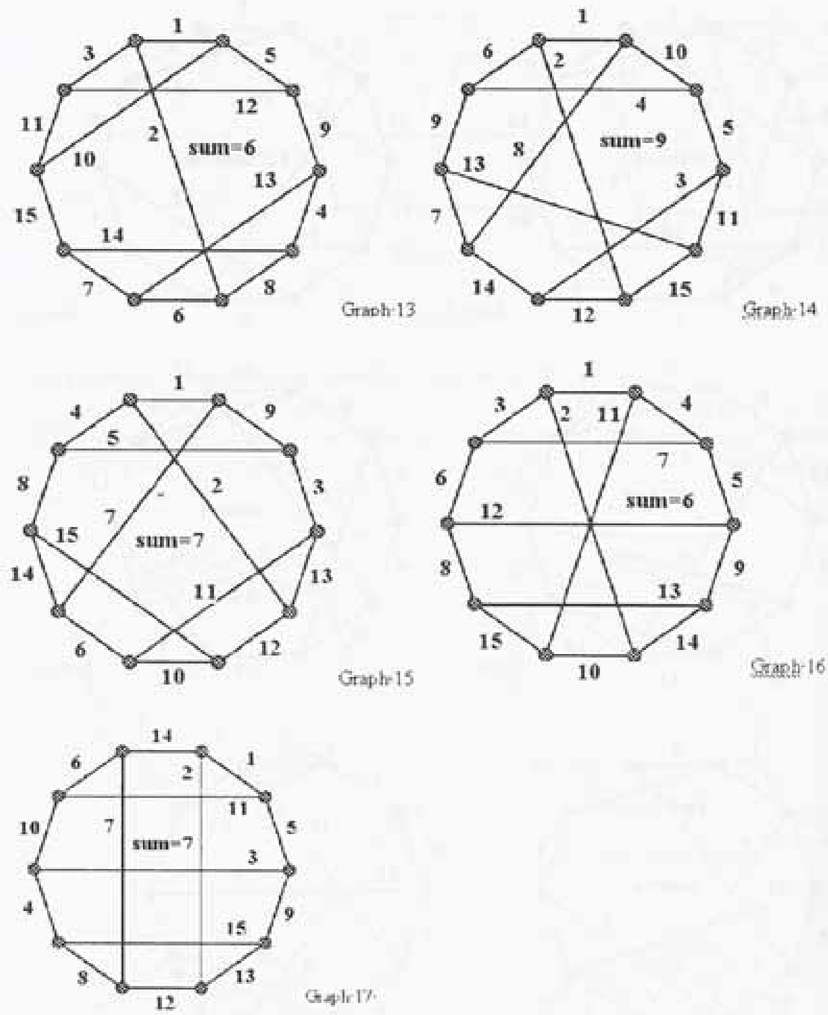


Figure 19.

Theorem 8. The connected simple cubic graph G_2 is the smallest non edge-magic graph which satisfies the necessary condition of edge-magic graph but not edge-magic.

That Graph 2 is not edge-magic is proved by a computer exhaustive search. It is quite time consuming. It took us more than three weeks and examined over 1.3 trillion labelings to show that no edge-magic labeling exists for Graph 2. Independently, Medei Kitagaki and Joseph Young also showed that Graph 2 is not edge-magic. It would be gratifying if the reader can provide a direct proof.

7. Some Conjectures.

The main contribution of this paper is to show that the conjecture of Lee is false. The question remaining unsolved is under what conditions is a cubic simple graph edge-magic?

We conclude this paper by proposing here several conjectures and invite the reader to solve them.

Conjecture 1 The graph $K_4+nK_2[3]$ is edge-magic for all odd $n \geq 9$.

Conjecture 2. For any $n \geq 2$, and any two non-parallel edges e_1,e_2 of $C_{2n} \times K_2$, the cubic graph $(C_{2n} \times K_2)\ \Delta 1\{e_1,e_2\}$ is edge-magic.

Conjecture 3. For any $n \geq 2$, and any two non-parallel edges e_1,e_2 of $C_{2n+1} \times K_2$, the cubic graph $(C_{2n+1} \times K_2)\ \Delta 2\{e_1,e_2\}$ is edge-magic.

Conjecture 4. We can add 2n new vertices and n edges to $K_2[3]$ as in Figure 15 and Figure 16 to form a series of edge-magic cubic graphs.

Conjecture 5. For any even integer n, we can add 2n new vertices and n edges to M(2k+1) as in Figure 18 to form a series of edge-magic cubic graphs.

Conjecture 6. Almost all simple cubic graphs of the form p =2(mod 4) are edge-magic.

References

[1] J.A. Gallian, A dynamic survey of graph labeling, *The Electronic J. of Combin.* (2001), # DS6, 1-79.

[2] Richard Guy and F. Harary, On the Mobius ladders, *Canadian Math Bulletin* **10**(1967) 493-496.

[3] N. Hartsfield and G. Ringel, Supermagic and antimagic graphs, *Journal of Recreational Mathematics*, **21** (1989), 107-115.

[4] Jonathan Keene and Andrew Simoson, Balanced strands for asymmetric, edge-graceful spiders, *Ars Combinatoria* **42** (1996),49-64..

[5] Y.S. Ho and Sin-Min Lee, Some initial results on the supermagicness of regular complete k-partite graphs, *The Journal of Combinatoric Mathematics and Combinatoric Computing* **39** (2001),3-17.

[6] Li Min Lee, Sin Min Lee, and G. Murty, On edge-graceful labelings of complete graphs - solutions of Lo's conjecture, *Congressum Numerantium* **62** (1988), 225-233.

[7] Sin-Min Lee, A conjecture on edge-graceful trees, *Scientia*, Ser. A, vol.**3**, (1989), 45-57.

[8] Sin- Min Lee, New Directions in the Theory of Edge-Graceful Graphs, *Proceedings of the 6th Caribbean Conference on Combinatorics & Computing*, (1991) 216-231..

[9] Sin-Min Lee, W.M. Pigg and T.J. Cox, On edge-magic cubic graphs conjecture, *Congressus Numerantium*, **105** (1994), 214-222..

[10] Sin Min Lee and Eric Seah, Edge-graceful labelings of regular complete k-partite graphs, *Congressum Numerantum* **75** (1990), 41-50.

[11] Sin-Min Lee, E. Seah and S.P. Lo , On edge-graceful 2-regular graphs, *The Journal of Combinatoric Mathematics and Combinatoric Computing*, **12** (1992), 109-117,.

[12] Sin Min Lee, Eric Seah and S.K. Tan, On edge-magic graphs, *Congressus Numerantium,* **86** (1992), 179-191.

[13] Sin-Min Lee, E. Seah, Siu-Ming Tong, On the edge-magic and edge-graceful total graphs conjecture, *Congressus Numerantium* 141(1999), 37-48.

[14] S.P. Lo, On edge-graceful labelings of graphs, *Congressus Numerantium,* **50** (1985) 231-241.

[15] Karl Schaffer and Sin Min Lee, Edge-graceful and edge-magic labelings of Cartesian products of graphs, *Congressus Numerantium* **141 (**1999) 119-134.

[16] W.C. Shiu, P.C.B. Lam and H.L.Cheng, Supermagic labeling of $sK_{n,n}$ *Congressus Numerantium* **146** (2000),119-124.

[17] W.C. Shiu, P.C.B. Lam and Sin-Min Lee, Edge-magicness of the composition of a cycle with a null graph, *Congressus Numerantium,* **132** (1998). 9-18,

[18] W.C. Shiu, P.C.B. Lam and Sin-Min Lee, On a construction of supermagic graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing,* **42** (2002) , 147-160

[19] W.C.Shiu and Sin-Min Lee, Some edge-magic cubic graphs, Journal of Combinatorial Mathematics and Combinatorial Computing, **40** (2002),115-127

[20] B.M. Stewart, Magic graphs, *Canadian Journal of Mathematics,* **18**(1966),

[21] B.M. Stewart, Supermagic complete graphs, *Canadian Journal of Mathematics,* **19**(1967), 427-438.