INTEGER-MAGIC SPECTRA OF AMALGAMATIONS OF STARS AND CYCLES

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Abstract. For any positive integer k, a graph $G = (V, E)$ is said to be $\mathbb{Z}_k$-magic if there exists a labeling $l : E(G) \longrightarrow \mathbb{Z}_k - \{0\}$ such that the induced vertex set labeling $l^+ : V(G) \longrightarrow \mathbb{Z}_k$ defined by

$$l^+(v) = \sum\{\ l(uv)\ :\ uv \in E(G)\ \}$$

is a constant map. For a given graph G, the set of all $h \in \mathbb{Z}_+$ for which G is $\mathbb{Z}_h$-magic is called the integer-magic spectrum of G and is denoted by $IM(G)$. In this paper, we will determine the integer-magic spectra of the graphs which are formed by the amalgamation of stars and cycles. In particular, we will provide examples of graphs that for a given $n > 2$, they are not h-magic for all values of $2 \leq h \leq n$.

1. Introduction

For any abelian group A, written additively, any mapping $l : E(G) \longrightarrow A - \{0\}$ is called a *labeling*. Given a labeling on the edge set of G one can introduce a vertex set labeling $l^+ : V(G) \longrightarrow A$ as follows:

$$l^+(v) = \sum\{\ l(uv)\ :\ uv \in E(G)\ \}.$$

A graph G is said to be A-*magic* if there is a labeling $l : E(G) \longrightarrow A - \{0\}$ such that for each vertex v, the sum of the labels of the edges incident with v are all equal to the same constant; that is, $l^+(v) = c$ for some fixed $c \in A$. We will call $< G, l >$ an A-magic graph with sum c. In general, a graph G may admit more than one labeling to become an A-magic graph; for example, if $|A| > 2$ and $l : E(G) \longrightarrow A - \{0\}$ is a magical labeling of

G with sum c, then $\lambda : E(G) \longrightarrow A-\{0\}$, the inverse labeling of l, defined by $\lambda(uv) = -l(uv)$ will provide another magical labeling of G with sum $-c$.

The original concept of A-magic graph is due to J. Sedlacek [?, ?], who defined it to be a graph with a real-valued edge labeling such that

(1) distinct edges have distinct nonnegative labels; and
(2) the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

Given a graph G, the problem of deciding whether G admits a magic labeling is equivalent to the problem of deciding whether a set of linear homogeneous Diophantine equations has a solution [?]. At present, given an abelian group, no general efficient algorithm is known for finding magic labelings for general graphs.

When $A = \mathbb{Z}$, the $\mathbb{Z}$-magic graphs were considered in Stanley [?, ?], he pointed out that the theory of magic labeling can be put into the more general context of linear homogeneous diophantine equations. When the group is $\mathbb{Z}_k$, we shall refer to the $\mathbb{Z}_k$-magic graph as *k-magic*. Graphs which are k-magic had been studied in [?, ?, ?, ?, ?, ?, ?]. For convenience, we will use the notation 1-magic instead of $\mathbb{Z}$-magic. Doob [?, ?, ?], also considered A-magic graphs where A is an abelian group. He determined which wheels are $\mathbb{Z}$-magic.

A graph $G = (V, E)$ is called *fully magic* [?, ?, ?, ?, ?, ?] if it is A-magic for every abelian group A, and it is called *non-magic* if for every abelian group A it is not A-magic. Also, a graph G is said to be $\mathbb{N}$-magic if there exists a labeling $l : E(G) \rightarrow \mathbb{N}$ such that $l^+(v)$ is a constant, for every $v \in V(G)$. It is well-known that a graph G is $\mathbb{N}$-magic if and only if each edge of G is contained in a 1-factor (a perfect matching) or a $\{1, 2\}$-factor [?, ?, ?]. Berge [?] called a graph *regularisable* if a regular multigraph could be obtained from G by adding edges parallel to the edges of G. In fact, a graph is regularisable if and only if it is $\mathbb{N}$-magic. For $\mathbb{N}$-magic graphs, readers are referred to [?, ?, ?, ?, ?, ?, ?, ?]. The notion of $\mathbb{Z}$-magic is weaker than $\mathbb{N}$-magic. Figure ?? shows a graph which is $\mathbb{Z}$-magic but not $\mathbb{N}$-magic.

Observation 1.1. *For any $n \geq 3$, the path of order n is non-magic.*

Observation 1.2. *In any magical labeling of a cycle the edges should alternatively be labeled the same group elements.*

Proof. Let $l : E(C_n) \rightarrow A$ be a magic labeling and e_1, e_2, e_3, e_4 be four consecutive edges of C_n. Then $l(e_1)+l(e_2) = l(e_2)+l(e_3)$ and $l(e_2)+l(e_3) = l(e_3) + l(e_4)$. Which implies that $l(e_1) = l(e_3)$ and $l(e_2) = l(e_4)$. □

Observation 1.3. *C_{2n}, the cycle of order $2n$, with a pendant edge is non-magic.*

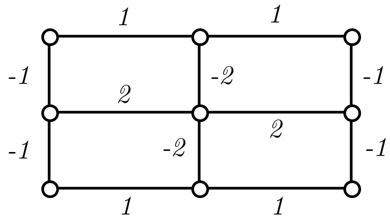

FIGURE 1. The graph $P_3 \times P_3$ is $\mathbb{Z}$-magic but is not $\mathbb{N}$-magic.

Proof. By the Observation **??**, it is enough to prove the statement for C_4. As illustrated in the Figure **??**, in any labeling, the sum of labels of the edges incident with vertex v needs to be equal to the sum of labels of the edges incident with u; that is, $x + y + z = x + y \Longrightarrow z = 0$, a contradiction. □

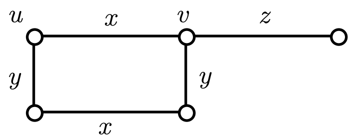

FIGURE 2. The cycle C_4 with an edge pendant is non-magic.

In this paper, we will denote the set of positive integers by $\mathbb{N}$, and for any $k > 0$,

$$k\mathbb{N} = \{\ kn\ :\ n \in \mathbb{N}\ \}, \quad \text{also} \quad k + \mathbb{N} = \{\ k + n\ :\ n \in \mathbb{N}\ \}.$$

For a given graph G the set of all positive integers h for which G is $\mathbb{Z}_h$-magic (or simply h-magic) is called the *integer-magic spectrum* of G and is denoted by $IM(G)$. Since any regular graph is fully magic, then it is h-magic for all positive integers $h \geq 2$; therefore, $IM(G) = \mathbb{N}$. For more general results on integer-magic spectrum of graphs, the reader is referred to [**?**, **?**, **?**].

A graph G with a fixed vertex u will be denoted by the ordered pair (G, u). Given two ordered pairs (G, u) and (H, v), one can form a new graph by amalgamation: form the disjoint union of G and H and identify u and v. The resulting graph will be denoted by $(G, u) \circ (H, v)$.
For convenience the complete bipartite graph $K(1, m)$, known as star with m leaves, will be denoted by $ST(m)$. Given a star $ST(m)$ and a cycle C_n, depending on whether we identify the center of the star with a vertex of C_n or identify an end-vertex of the star with a vertex of C_n, the amalgamation of these two graphs will result in two non-isomorphic graphs. We will

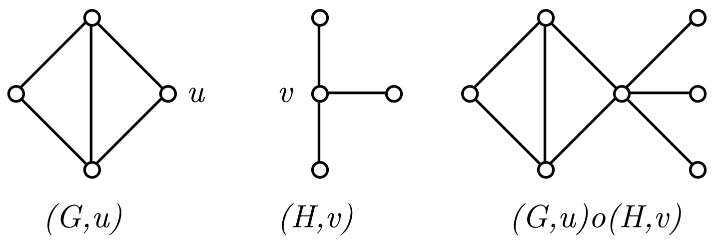

FIGURE 3. Amalgamation Construction.

denote the first one by $ST(m)\#C_n$, and the latter by $ST(m)@C_n$. Figure **??** illustrates the two different amalgamations of $ST(3)$ and C_4.

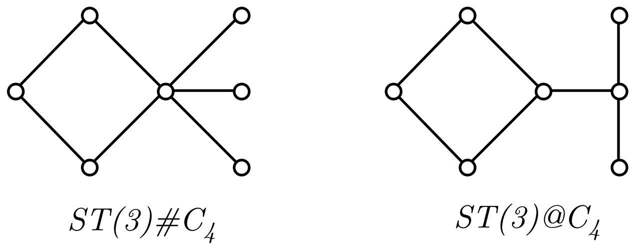

FIGURE 4. Two different amalgamations of $ST(3)$ and C_4.

2. Integer-Magic Spectrum of $ST(m)\#C_n$.

By the Observation **??**, in any magic labeling of C_n, labels of the edges alternates. One immediate consequence of this fact is that, the graph $ST(m)\#C_3$ (or $ST(m)\#C_4$) is h-magic if and only if the graph $ST(m)\#C_{2k+1}$ (or $ST(m)\#C_{2k}$) is h-magic. As a result, we will only concentrate on cycles C_3 and C_4. Also, we observe that since $\{1, 2\}$ is a subset of the degree set of these types of graphs, $ST(m)\#C_n$ can not be 2-magic.

Theorem 2.1. $IM(\,ST(1)\#C_4\,) = \emptyset$.

Proof. This is a direct result of the Observation **??**. □

Theorem 2.2. $IM(\,ST(2)\#C_4\,) = 2 + 2\mathbb{N}$.

Proof. We observe that $ST(2)\#C_4$ is h-magic if and only if $2|h$ and $h > 2$. Because, as illustrated in the Figure **??**, we need to have $l^+(w) = l^+(u)$ or $2z \equiv 0 \pmod{h}$, and this means z is an order two element of the group $\mathbb{Z}_h$. Therefore, $2|h$. On the other hand, if $h = 2r$, then the selection of $x = 1,\ y = r - 1,\ z = r$ will work. □

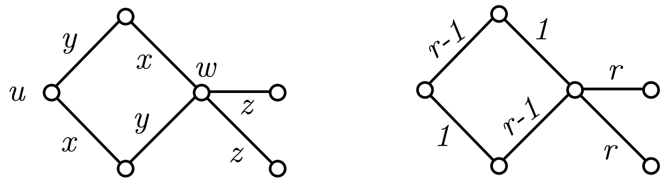

FIGURE 5. General labeling of $ST(m)\#C_4$. Here $z = x + y$.

Theorem 2.3. *For any positive integer $m \geq 2$, the graph $ST(m)\#C_4$ is h-magic if and only if $h > 2$ and $\gcd(m, h) > 1$. Therefore,*

$$IM(\, ST(m)\#C_4\,) = (2 + I\!N) - \{\ h \in I\!N\ :\ \gcd(m, h) = 1\ \}.$$

Proof. In $ST(m)\#C_4$ there are m pendant edges attached to the vertex w and the equation $l^+(w) = l^+(u)$ implies that

$$mz \equiv 0 \pmod{h}. \tag{2.1}$$

If $\gcd(m, h) = 1$, then the equation **??** will be equivalent to $z \equiv 0 \pmod{h}$, which does not provided a non-zero solution. Now suppose $\gcd(m, h) = \delta > 1$. If $\delta = h$, then we use $x = y = 1$ and $z = 2$ as our labels, with $l^+ \equiv 2$. Suppose $1 < \delta < h$, and let $h = \delta r$ $(r \geq 2)$. Then the selection $x = 1,\ y = r - 1$, and naturally $z = r$ will work with $l^+ \equiv r$. □

Corollary 2.4. *If p is a prime number bigger than 2, then*

$$IM(\, ST(p)\#C_4\,) = pI\!N.$$

Corollary 2.5. *If $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is an odd positive integer, then*

$$IM(\ ST(m)\#C_4\ ) = \bigcup_{i=1}^{k} (p_i I\!N).$$

If $m = 2^{\beta} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is even, then

$$IM(\ ST(m)\#C_4\ ) = (2 + 2I\!N) \bigcup_{i=1}^{k} (p_i I\!N).$$

Theorem 2.6. $IM(\, ST(1)\#C_3\,) = 2 + 2I\!N$.

Proof. For graph $ST(1)\#C_3$ to be h-magic, as illustrated in the Figure **??**, we need to have $2x + z = z$ or $2x \equiv 0 \pmod{h}$. That is, x is a non-zero element of $\mathbb{Z}_h$ with order 2. Therefore, $2|h$. On the other hand, if $h = 2r$, then the selections $x = r,\ y = 1$ will work; that is, the graph $ST(1)\#C_3$ has h-magic. □

Theorem 2.7. $IM(\, ST(2)\#C_3\,) = I\!N - \{2, 3\}$.

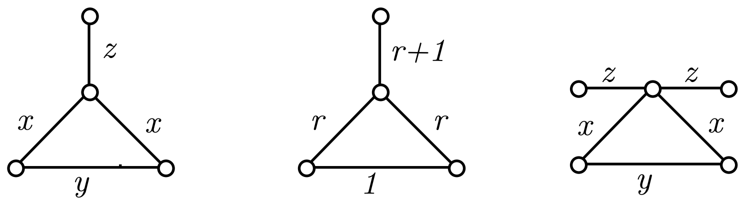

FIGURE 6. $ST(1)\#C_3$ and $ST(2)\#C_3$.

Proof. For the graph $ST(2)\#C_3$ to be h-magic, as illustrated in the Figure **??**, we need to have $2x+2z=z$ or $2x+z\equiv 0 \pmod h$. Here $z=x+y$, therefore we need $3x+y\equiv 0 \pmod h$, and this equation does not provide non-zero solutions for $h=3$. If $h>3$, then the selections $x=1,\ y=h-3$ will work. This means the graph is h-magic for all $h\geq 4$. Furthermore, $ST(2)\#C_3$ is $\mathbb{Z}$-magic; because, the labels $x=-1,\ y=3$ will work with $l^+\equiv 2$. $\square$

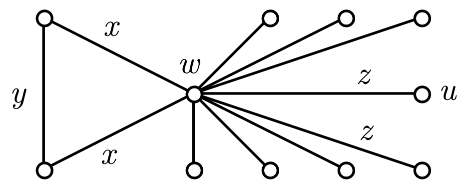

FIGURE 7. A typical labeling of $ST(m)\#C_3$. Here, $z=x+y$.

Note that when $m>1$, the graph $ST(m)\#C_3$ is $\mathbb{Z}$-magic; because, the labels $x=1-m,\ y=1+m$, and $z=2$ will work with $l^+\equiv 2$. In addition, any magical labeling of $ST(m)\#C_3$, as illustrated in the Figure **??**, uses three non-zero group elements x, y, and $z=x+y$. In particular, we need to have $l^+(u)=l^+(w)$, which implies

$$(m-1)z+2x\equiv 0 \pmod h, \tag{2.2}$$

or equivalently

$$(m+1)x+(m-1)y\equiv 0 \pmod h. \tag{2.3}$$

Theorem 2.8. *The graph $ST(m)\#C_3$ is 3-magic if and only if $3|m$.*

Proof. If $m=3k$, the labeling $x=y=1$ will work with $l^+\equiv 2 \pmod 3$. If $m=3k+1$, the equation **??** will become $(3k+2)x+3ky\equiv 0 \pmod 3$, which is equivalent to $2x\equiv 0 \pmod 3$, or $x\equiv 0$, which is not an acceptable answer. If $m=3k+2$, the equation **??** will become $(3k+3)x+(3k+1)y\equiv 0 \pmod 3$, which is equivalent to $y\equiv 0$, not an acceptable answer. $\square$

Theorem 2.9. *For every $h > m+1$, the graph $ST(m)\#C_3$ is h-magic.*

Proof. If $h > m+1$, then we choose $x = h-(m-1)$, $y = m+1$, and naturally $z = 2$. We notice that x, y, z are non-zero elements of $\mathbb{Z}_h$ and $l^+(w) = mz + 2x = 2m + 2(h-m+1) = 2h+2 \equiv 2 \pmod{h}$. □

Theorem 2.10. *If $m > 2$, then the graph $ST(m)\#C_3$ is m-magic.*

Proof. The labeling $x = y = 1$ works with $l^+ \equiv 2$. □

Theorem 2.11. *The graph $ST(m)\#C_3$ is h-magic for all even positive integers $h \geq 4$.*

Proof. To prove the theorem it is enough to show that the equation **??** has two distinct non-zero solutions for z and x in $\mathbb{Z}_h$. Let $h = 2^\alpha \mu$, where μ is an odd number, and consider the equations

$$(m-1)z + 2x \equiv 0 \pmod{2^\alpha}, \tag{2.4}$$

$$(m-1)z + 2x \equiv 0 \pmod{\mu}. \tag{2.5}$$

Given any z, the equation **??** has the solution $x \equiv -\bar{2}\,(m-1)z$ for x, where $\bar{2}$ is the multiplicative inverse of 2 in $\mathbb{Z}_\mu$. For the equation **??** we consider two cases:

Case 1. When $m = 2k+1$ is odd, then the equation **??** becomes $2kz + 2x \equiv 0 \pmod{2^\alpha}$ or $kz + x \equiv 0 \pmod{2^{\alpha-1}}$, and for any z we will have $x \equiv -kz \pmod{2^{\alpha-1}}$. Therefore, $x \equiv -kz \pmod{2^{\alpha-1}\cdot\mu}$, and for any non-zero $z \in \mathbb{Z}_h$, one can choose either $x \equiv -kz$ or $x \equiv -kz + 2^{\alpha-1}\cdot\mu \pmod{h}$ for the solution of the equation **??**.

Case 2. When $m = 2k$ is even, then z has to be even. Let $z = 2\xi$. Equation **??** becomes $(m-1)\xi + x \equiv 0 \pmod{2^{\alpha-1}}$, and for any ξ we will have $x \equiv -(m-1)\xi \pmod{2^{\alpha-1}}$. Therefore, $x \equiv -(m-1)\xi \pmod{2^{\alpha-1}\cdot\mu}$, and for any non-zero $z = 2\xi \in \mathbb{Z}_h$, one can choose either $x \equiv -(m-1)\xi$ or $x \equiv -(m-1)\xi + 2^{\alpha-1}\cdot\mu \pmod{2^\alpha}$ for the solution of the equation **??**.

□

As an application of the Theorem **??**, we will show that $ST(m)\#C_3$ is always 4-magic. Using the notation and the process of the theorem, we will consider two cases:

Case 1. If $m = 2k+1$, then we need $kz + x \equiv 0 \pmod{2}$ or $x \equiv -kz \pmod{2}$. For non-zero $z = 3$, we have two choices of either $x \equiv -3k \pmod{4}$ or $x \equiv -3k+2 \pmod{4}$, which translates to either $x = 1$ or $x = 2$.

Case 2. If $m = 2k$, then we will deal with the equation $(2k-1)z + 2x \equiv 0 \pmod 4$, which implies that $z = 2\xi$. With this consideration the equation becomes $(2k-1)\xi + x \equiv 0 \pmod 2$ or $x \equiv \xi \pmod 2$. Now for the only choice of $z = 2$ (or $\xi = 1$) we will have two choices of either $x \equiv 1$ or $x \equiv 3$ and the corresponding values of y will be 1 or 3, respectively.

Theorem 2.12. *Let $h \geq 3$ be an odd positive integer. Then $ST(m)\#C_3$ is h-magic if and only if h is not a divisor of $m+1$ and $m-1$.*

Proof. Suppose h is a divisor of $m+1$ or $m-1$. Since $\gcd(m+1, m-1) = 1$ or 2 and h is odd, then h can only be divisor of one of them. Without loss of generality, we may assume that h is an odd divisor of $m+1$. As a result $\gcd(h, m-1) = 1$, and equation **??**, $(m+1)x + (m-1)y \equiv 0 \pmod h$, becomes $y \equiv 0 \pmod h$, that does not provide non-zero solution for y.
Conversely, assume that h is not a divisor of $m-1$ and $m+1$, and let $m = hq + r$. Then $r \neq \pm 1$; otherwise, one of $m-1$ or $m+1$ will be divisible by h. As a result, the selections of $x = d+1-r$ and $y = 1+r$ are valid and will work with $l^+(w) = mz + 2x = 2(hq+r) + 2(d+1-r) \equiv 2 \pmod h$. □

We conclude the section by the following theorem, which is the natural consequence of the Theorems **??** through **??**. This theorem will completely determine the integer-magic spectrum of $ST(m)\#C_3$.

Theorem 2.13. *If $m \geq 2$, then the integer-magic spectrum of the graph $ST(m)\#C_3$ is*

$$I\!N - \{\ d \in I\!N\ :\ d = 2 \text{ or } d \text{ is an odd divisor of } m+1 \text{ or } m-1\ \}.$$

As an application of this theorem, consider the graph $ST(134)\#C_{23}$. To find its integer-magic spectrum it is enough to consider $ST(134)\#C_3$, where $m = 134$. Now the odd divisors of $m+1 = 135$ and $m-1 = 133$ are $3, 5, 7, 9, 15, 19, 27, 45, 133$, and 135. Therefore,

$$IM(\,ST(134)\#C_{23}\,) = I\!N - \{\ 2, 3, 5, 7, 9, 15, 19, 27, 45, 133, 135\}.$$

3. Integer-Magic Spectrum of $ST(m)@C_n$.

By the Observation **??**, in any magic labeling of C_n, the labels of the edges alternates. One immediate consequence of this fact is that, the graph $ST(m)@C_3$ (or $ST(m)@C_4$)is h-magic if and only if the graph $ST(m)@C_{2k+1}$ (or $ST(m)@C_{2k}$) is h-magic. As a result, we will only concentrate on cycles C_3 and C_4. Also, we observe that since $\{1, 2\}$ is a subset of the degree set of these types of graphs, they can not be 2-magic. In general any magic labeling of $ST(m)@C_3$, as illustrated in the Figure **??**, uses three non-zero distinct group elements x, y, and $z = x + y$. Therefore, for any abelian group A, a necessary condition for $ST(m)@C_3$ to be A-magic is that $|A| \geq 4$. Hence, for any $m \in I\!N$, the graph $ST(m)@C_3$ is not

3-magic. Moreover, when $m \geq 4$, then $ST(m)@C_3$ is $\mathbb{Z}$-magic; because, the labels $x = m-1,\ y = 3-m$ work with $l^+ \equiv 2$. Therefore, the integer-magic spectrum of these graphs will be contained in $\mathbb{N} - \{2,3\}$.

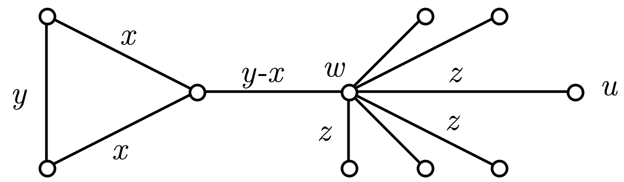

FIGURE 8. A typical magical labeling of $ST(m)@C_3$. Here $z = x + y$.

Theorem 3.1. *$IM(\,ST(m)@C_4\,) = \emptyset$ for every $m \geq 1$.*

Proof. This is a direct result of the Observation **??**. □

We observe that $ST(1)@C_3 = ST(1)\#C_3$, therefore $IM(\,ST(1)@C_3\,) = 2 + 2\mathbb{N}$. Also, by **??**, $ST(2)@C_3$ is non-magic, or $IM(\,ST(2)@C_3\,) = \emptyset$. To determine the integer-magic spectrum of $ST(m)@C_3$, from now on we will assume that $m \geq 3$. Also, in any magic labeling of $ST(m)@C_3$ one needs to have $l^+(w) = l^+(u)$ or $(m-1)z + y - x = z$, which implies

$$(m-3)x + (m-1)y \equiv 0 \pmod{h}, \tag{3.1}$$

or equivalently $(z = x + y)$

$$(m-1)z - 2x \equiv 0 \pmod{h}, \tag{3.2}$$

Theorem 3.2. *$IM(ST(3)@C_3) = 2 + 2\mathbb{N}$.*

Proof. When $m = 3$, the equation **??** will become $2y \equiv 0 \pmod{h}$. This means that y is a member of $\mathbb{Z}_h$, which has order 2, therefore $2|h$ and h is even. On the other hand, whenever $h = 2r$, the graph is h-magic, as illustrated in the Figure **??**. □

Theorem 3.3. *If $m \geq 4$, then the graph $ST(m)@C_3$ is $\mathbb{Z}$-magic.*

Proof. We observe that the choices of $x = m-1,\ y = -m+3$ and $z = 2$ will give us three distinct non-zero integers with $l^+(w) = (m-1)z + y - x = 2(m-1) - m + 3 - m + 1 = 2$. □

Theorem 3.4. *For any abelian group A, if $|A| \leq 4$, then $ST(4)@C_3$ is not A-magic. Furthermore, $IM(ST(4)@C_3) = \mathbb{N} - \{2,3,4\}$.*

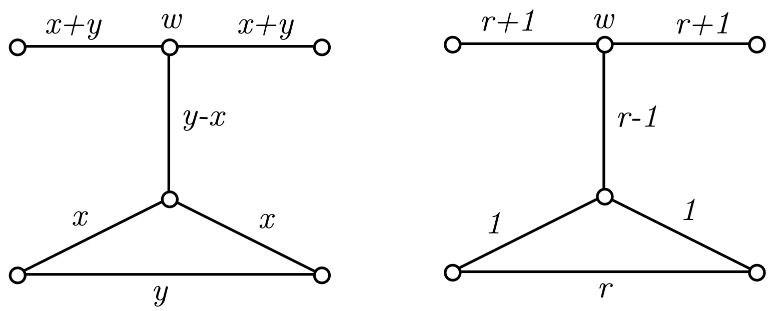

FIGURE 9. $IM(ST(3)@C_3) = 2 + 2\mathbb{N}$.

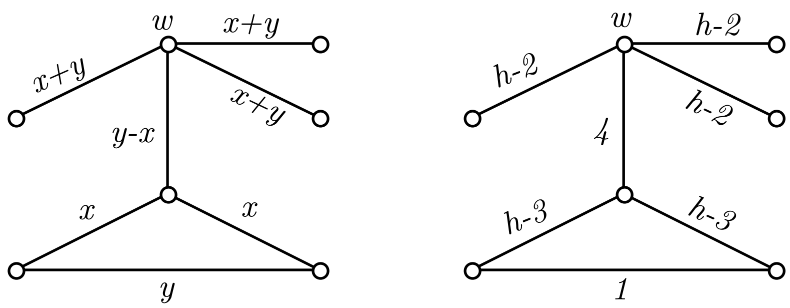

FIGURE 10. $IM(ST(4)@C_3) = \mathbb{N} - \{2, 3, 4\}$.

Proof. Any magic labeling of this graph, as illustrated by the Figure **??**, will require three distinct non-zero elements of the abelian group A; namely, x, y, and $x + y$. Also, in this case, equation **??** becomes $x + 3y = 0 \pmod{h}$. This implies that $x + 2y$ is another non-zero element of this group other than x, y, and $x+y$. Therefore, the group A must have at least five elements. Furthermore, as illustrated in the Figure **??**, the graph is h-magic for every $h \geq 5$. □

Corollary 3.5. *The graph $ST(m)@C_3$ is not 3-magic.*

Theorem 3.6. *For any $m \geq 5$ the graph $ST(m)@C_3$ is m-magic.*

Proof. The choices of $x = m - 1$, $y = 3$, and $z = 2$ will provide three distinct non-zero elements of $\mathbb{Z}_m$ with $l^+(w) = 2$. □

Theorem 3.7. *If m is an odd positive integer, then the graph $ST(2k + 1)@C_3$ is 4-magic.*

Proof. We consider two cases:

Case 1. If $m = 4k + 1$, then the choices of $x = 2,\ y = 3$, and $z = 1$ will work with $l^+(w) = 4k + 1 \equiv 1 \pmod 4$.

Case 2. If $m = 4k + 3$, then the choices of $x = 3,\ y = 2$, and $z = 1$ will work with $l^+(w) = (4k + 2) + 3 \equiv 1 \pmod 4$.

□

Theorem 3.8. *If $m = 2k + 1 \geq 5$, then the integer-magic spectrum of $ST(2k + 1)@C_3$ is*

$$\mathbb{N} - \{2\} \cup \{\ d > 1\ :\ d \text{ is an odd divisor of one of } m-1,\ m-2,\ \text{or } m-3\ \}.$$

Proof. We will prove the theorem in five steps:

Step 1. In this part we will show that for any $h \geq 2k$ the graph $ST(2k + 1)@C_3$ is h-magic. Because, the labeling of $x = k,\ y = h + 1 - k$, and naturally, $z = 1$ will work. Here we note that $y - x = h + 1 - 2k \not\equiv 0 \pmod h$ and $l^+(w) = 2kz + y - x \not\equiv 1 \pmod h$.

Step 2. If h is any divisor of $m - 2$, then $ST(m)@C_3$ is not h-magic. Because, the equation **??**, is equivalent to $(m-2)z+y-x \equiv 0 \pmod h$, and since h is a divisor of $m - 2$, we will have $y - x \equiv 0 \pmod h$, which is not an acceptable solution.

Step 3. If h is an odd divisor of either $m - 1 = 2k$ or $m - 3 = 2k - 2$, then $ST(2k + 1)@C_3$ is not h-magic. Because, the equation **??** becomes $(2k - 2)x + 2ky \equiv 0 \pmod h$ or $(k - 1)x + ky \equiv 0 \pmod h$. Since $\gcd(k - 1, k) = 1$, without loss of generality, we may assume that $d|(k - 1)$ and $\gcd(d, k) = 1$. As a result will get $y \equiv 0 \pmod h$, which is not an acceptable answer.

Step 4. If h is any odd number that is not a divisor of any one of $m - 1 = 2k,\ m - 2 = 2k - 1,\ m - 3 = 2k - 2$, then $ST(m)@C_3$ is h-magic. Because, the labels $x \equiv k, y \equiv 1 - k$, and naturally $z = 1$ are three non-zero distinct elements of $\mathbb{Z}_h$, will work with $l^+ \equiv 1$. Note that $y - x \equiv 1 - 2k \not\equiv 0 \pmod h$.

Step 5. If $4 < h \leq m - 1 = 2k$ is an even number, then $ST(2k + 1)@C_3$ is h-magic. Because, the equation **??** becomes $2kz - 2x \equiv 0 \pmod h$, which is equivalent to $kz - x \equiv 0 \pmod{h/2}$. Now for any non-zero $z \in \mathbb{Z}_h$, we have two choices for x; namely, $x \equiv kz \pmod h$, or $x \equiv kz + \frac{h}{2} \pmod h$.

□

Examples 3.9.

(a) $IM(\,ST(5)@C_3\,) = \mathbb{N} - \{2, 3\}$. Here, $m = 5$. We need to exclude 2 and the odd divisors of $m - 1 = 4,\ m - 2 = 3$, and $m - 3 = 2$.

(b) $IM(\,ST(7)@C_3\,) = \mathbb{N} - \{2, 3, 5\}$. Here, $m = 7$. We need to exclude 2 and the odd divisors of $m - 1 = 6,\ m - 2 = 5$, and $m - 3 = 4$.

(c) $IM(\, ST(45)@C_3\,) = I\!N - \{2,3,7,11,21,43\}$. Here, $m = 45$. We need to exclude 2 and the odd divisors of $m-1 = 44$, $m-2 = 43$, and $m-3 = 42$.

(d) $IM(\, ST(135)@C_{95}\,) = I\!N - \{2,3,7,11,19,33,67,133\}$.

Theorem 3.10. *If $m = 2k \geq 4$, then the integer-magic spectrum of $ST(m)@C_3$ is*

$$I\!N - \{\ h > 1\ :\ h|(2m-4),\ \textit{or}\ \ h|(m-1),\ \textit{or}\ \ h|(m-3)\ \}.$$

Proof. We will prove the theorem in three steps:

Step 1. If $h > 2m-4$, then $ST(m)@C_3$ is h-magic. Because, the choices of $x = m-1$, $y = h-m+3$, and naturally $z = x+y = 2$, will work with $l^{+}(w) = 2(m-1)+4-2m = 2$. Also, note that $y - x = 4 - 2m \not\equiv 0 \pmod h$.

Step 2. If $h > 1$ is any divisor of $2m-4$, then the graph $ST(m)@C_3$ is not h-magic. It is enough to prove this statement for $h > 2$. Let $2m-4 = hq$, which implies that $m-1 = -(m-3)+hq$ or $m-1 \equiv -(m-3) \pmod h$. In this case the equation **??** becomes $(m-3)(x-y) \equiv 0 \pmod h$. Since $\gcd(m-3, 2m-4) = 1$ or 2, we have $\gcd(h, m-3) = 1$, as a result $x \equiv y \pmod h$, which does not provide a valid labeling.

Step 3. If h is an odd divisor of either $m-1$ or $m-3$, then $ST(m)@C_3$ is not h-magic. Here we observe that since $\gcd(m-3, m-1) = 1$ or 2 and h is odd, then h cannot divide both $m-3$ and $m-1$. Without loss of generality, we may assume that h is a divisor of $m-3$ and $\gcd(h, m-1) = 1$. In this case, the equation **??** becomes $y \equiv 0 \pmod h$, which does not provide a valid solution.

□

Examples 3.11.

(a) $IM(\, ST(4)@C_3\,) = I\!N - \{2,3,4\}$. Here, $m = 4$, and we need to exclude all the divisors $d > 1$ of $2m-4 = 4$, $m-1 = 3$, and $m-3 = 1$.

(b) $IM(\, ST(6)@C_3\,) = I\!N - \{2,3,4,5,8\}$. Here, $m = 6$. We need to exclude all the divisors of $2m-4 = 8$, $m-1 = 5$, and $m-3 = 3$.

(c) $IM(\, ST(14)@C_3\,) = I\!N - \{2,3,4,6,8,11,12,13,24\}$. Here, $m = 14$. We need to exclude all the divisors of $2m-4 = 24$, $m-1 = 13$, and $m-3 = 11$.

(d) The integer-magic spectrum of $ST(38)@C_{95}$ is

$$I\!N - \{h \in I\!N\ :\ 2 \leq h \leq 9\} \cup \{12,18,24,35,36,37,72\}.$$

Theorem 3.12. *Given any $n \geq 2$, there is a graph G such that G is not h-magic for every $h = 2,3,\cdots,n$.*

Proof. As we observed in the **??**, the graph $ST(4)@C_3$ is not h-magic for $h = 2, 3, 4$. So we may assume that $n \geq 4$. Let μ be the least common multiple of the numbers $2, 3, \cdots, n$, and consider the graph $G = ST(m)@C_3$, where $m = \dfrac{\mu + 4}{2}$. Note that here μ is divisible by 4, and so is $\mu + 4$, which implies that m is even. Also, any $h = 2, 3, \cdots, n$, is a divisor of $\mu = 2m - 4$. Therefore, the graph $ST(m)@C_3$ is not h-magic. □

We note that the number m, presented in the proof of the theorem **??**, might not be the smallest possible answer. For example, in **??**, we realized that $ST(38)@C_3$ is not h-magic for all $2 \leq h \leq 9$, while it is 10-magic. In this case, number 38 works, while the least common multiple of the numbers $2, 3, \cdots, 9$ is $\mu = 2520$, and the number provided by this theorem is $m = 1262$.
We conclude this paper by the following problems:

Problem 3.13. *For any positive integer $n \in \mathbb{Z}_+$ find the smallest $m \in \mathbb{N}$ such that the graph $ST(m)@C_3$ is not h-magic for all $2 \leq h < n$.*

Problem 3.14. *As examined in **??**, the graph $ST(6)@C_3$ has the property that it is 6-magic but it is not h-magic for all $2 \leq h \leq 5$. Find all $m \in \mathbb{N}$ such that $ST(m)@C_3$ is not h-magic, for all $2 \leq h < m$.*

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