On Super Edge-Magic Graphs with Many Odd Cycles

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It's easy to have a complicated idea. It's very, very hard to have a simple idea.
-----Carver Mead

Dedicated to Professor W. Wallis

ABSTRACT. A (p, q) graph G is *total edge-magic* if there exits a bijection f: $V \cup E \to \{1, 2, \ldots, p + q\}$ such that f(u) + f(e) + f(v) is a constant independent of e = (u, v) in E . A total edge-magic graph is called *super edge-magic* if f(V(G)) = {1, 2, …, p}. The super edge-magic properties of several classes of graphs with many odd cycles are studied.

1. Introduction.

In this paper we consider graphs with no loops. For undefined concepts we refer the reader to [3].

A (p, q)-graph G = (V, E) with p vertices and q edges is called ***total edge magic*** if there is a bijection $f : V \cup E \to \{1, 2, \ldots, p + q\}$ such that there exists a constant s for any (u, v) in E we have f(u) + f((u, v)) + f(v) = s. The original concept of total edge-magic graph is due to Kotzig and Rosa [17, 18]. They called it magic graph. A total edge-magic graph is called a ***super edge-magic*** if f(V(G)) = {1, 2, …, p}. Figure 1 shows a graph with a total edge-magic labeling. Figure 2 provides a super edge-magic labeling of this graph.

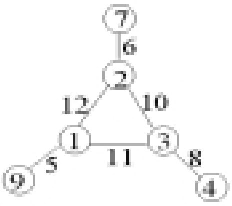


Figure 1.

A subset S of integers is called ***consecutive*** if S consists of consecutive integers. Chen [4] showed that a graph G is super edge-magic if and only if there exists a vertex labeling f such that the two sets f(V(G)) and $\{f(u) + f(v):(u, v) \in E(G)\}$are both consecutive. Independently, Figueroa -Centeno et al [8, 10] also obtained the same result. They showed that if f: $V(G) \to \{1, 2, \ldots, p\}$ is a bijection of a (p, q)-graph G and $S = \{f(u) + f(v): uv \in E\}$ is consecutive with $s = \min(S)$, then f can be extended to a super edge-magic labeling of G defined by $f(uv) = p + q + s - f(u) - f(v)$ for all edge uv of E(G) (Figure 2). In light of this result, it suffices to exhibit the vertex labeling of a super edge-magic graph labeling and shows it induces a consecutive labeling on edges.

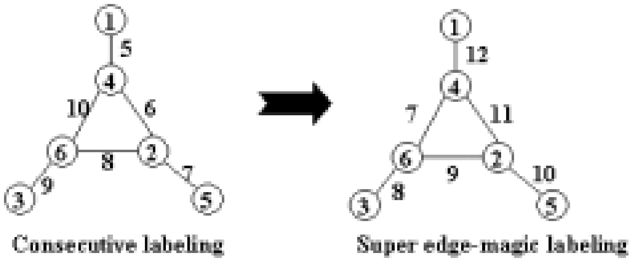


Figure 2.

Kotzig and Rosa [17, 18] proved that all cycles, complete bipartite graphs and caterpillars are total edge-magic. A complete graph K_n is total edge-magic if and only if $n \in \{1, 2, 3, 5, 6\}$ and the disconnected graph nK_2 is totally edge-magic if and only if n is odd. In fact, in [17] they showed that all caterpillars and $(2k + 1)K_2$ graph are super edge-magic. However, the edge-magic labeling of odd cycles [12] is not super edge-magic. In [6], Enomoto et al gave a super edge-magic labeling for odd cycles. Craft and Tesar [5] and independently Godbord and Slater [14] showed that all cycles are total edge-magic.

Grace [12] called a (p,q)-graph G **sequential** if there exists a labeling $f: V(G) \to \{0, 1, 2, \ldots, q - 1\}$ such that for some c the edge labels induced by $f(x)+f(y)$ for each edge (x,y) are $c, c + 1, c + 2, \ldots, c + q - 1$. Thus based on Chen's results we see that every super edge-magic graph is sequential.

In 1980, Graham and Sloane [13] introduced the concept of harmonious graphs. A graph with q edges is harmonious if there exist an injection $f: V(G) \to \{0, 1, 2, \ldots, q - 1\}$ such that the induced edge labeling $f^+: E(G) \to \{0, 1, 2, \ldots, q-1\}$ defined by $f^+(\{u, v\}) = f(u) + f(v) \pmod q$ is injective. Clearly all sequential graphs are harmonious.

In 1991, Lee, Schmeichel and Shee [20] introduced the concept of felicitous graphs. A graph with q edges is felicitous if there exist an injection $f: V(G) \to Z_{q+1}$ such that the induced edge labeling $f^+: E(G) \to \{0, 1, 2, \ldots, q - 1\}$ defined by $f^+(\{u, v\}) = f(u) + f(v) \pmod q$ is injective.

Cahit [2] defined a graph to be cordial if there exists a labeling $f: V(G) \to Z_2$ with an induced edge labeling $f(uv) = f(u) - f(v) \pmod 2$ such that if $v_f(i)$ and $e_f(i)$ are the number of vertices v and edges e satisfying that $f(v) = i$ and $f(e) = i$ for all $i \in Z_2$, respectively, then $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

The investigation of various graphs which are harmonious, felicitous or cordial has received significant attention. [1, 12, 15, 16, 21, 22, 25, 26, 27, 28, 29]. Figueroa et al [8] showed that if a (p, q)-graph G with $q \geq p$ is super edge-magic then G is harmonious and hence felicitous. Every super edge-magic graph is cordial.

A connected graph with one cycle is called unicyclic. We investigate some super edge-magic unicyclic graphs in [20]. In this paper we propose to investigate the existence of super edge-magic labelings for certain classes of graphs with many odd cycles. By means of these results we providenew classes of harmonious and cordial graphs.

For the other results of total edge-magic and super edge-magic graphs we refer to [7, 9, 11, 23, 30, 31, 32, 33, 34, 35].

2. Super edge-magic graphs with many odd cycles.

In [6,8], it was shown that the cycle C_n is super edge-magic if and only if n is odd. The complete bipartite graph $K_{m,n}$ is super edge-magic if and only if m = 1 or n = 1. We would like to consider graphs with many odd cycles.

Let $P_{2n}(+)N_m$ be the graph with $p=2n+m$ and $q = 2(m + n) - 1$.

$V(P_{2n}(+)N_m) = \{v_1, v_2, \ldots, v_{2n}, y_1, y_2, \ldots, y_m\}$ where $V(P_{2n}) = \{v_1, v_2, \ldots, v_{2n}\}$ and $V(N_m) = \{y_1, y_2, \ldots, y_m\}$.

$E(P_{2n}(+)N_m) = E(P_{2n}) \cup \{(v_1, y_1), (v_1, y_2), \ldots (v_1, \ldots, y_m), (v_{2n}, y_1), (v_{2n}, y_2), \ldots, (v_{2n}, y_m)\}$

The graph has m cycles of length 2n + 1 and many 4-cycles.

Theorem 1. $P_{2n}(+)N_m$ is super edge-magic for all $n, m \geq 1$.

Proof. Let us define $f: V(P_{2n}(+)N_m) \to \{1, 2, 3, \ldots, 2(m + n)-1\}$ as follows:

$f(v_{1+2t}) = 1 + t$, $t = 0, 1, \ldots, n - 1$

$f(v_{2+2t}) = m + n + 1 + t$, $t = 0, 1, \ldots, n - 1$

$f(y_k) = k + n$, $k = 1, 2, \ldots, m$

It can be easily checked that $f^+(E(P_{2n}(+)N_m))$ is a consecutive set.

Example 1. Super edge-magic labeling of $P_2(+)N_m$ and $P_4(+)N_m$ is depicted for m = 1, 2, 3 in Figure 3.

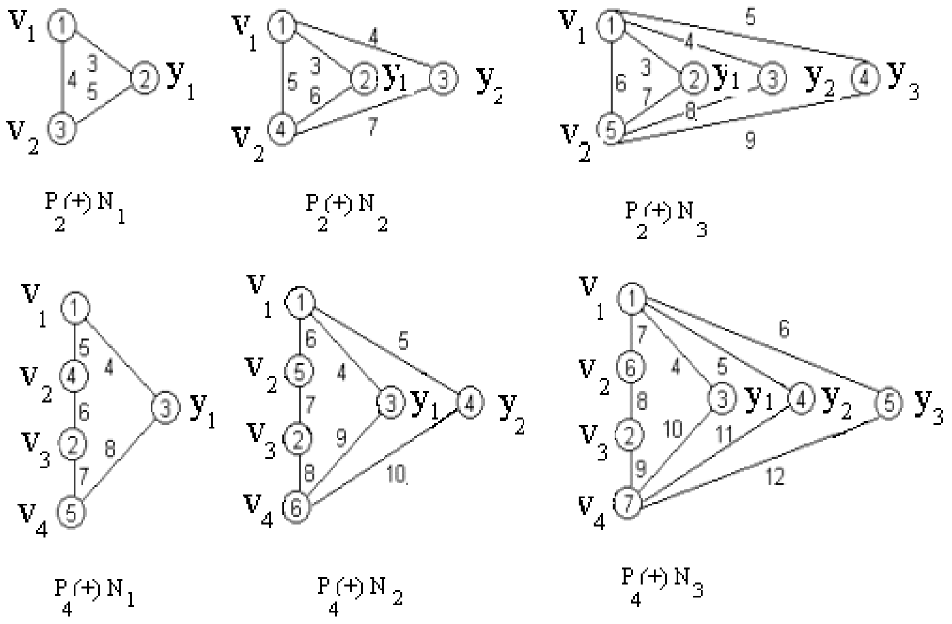


Figure 3.

Remark: Theorem 1 extends the result of Figueroa et al in [10]. They showed that $P_2(+)N_m$ is super edge-magic for all $m \geq 1$.

Enomoto et al [6] proved that if a (p, q)-graph G is super edge-magic, then $q \leq 2p - 3$ and it has at least two vertices of degree less than 4. Using this result, we can show that some wheel $C_n + K_1$ is total edge-magic but none of them can be super edge-magic.

Theorem 2. For $k \geq 1$, the planar graph $(P_2 \cup k K_1) + N_2$ is super edge-magic.

Proof. Let the vertex set of $P_2 \cup k K_1$ be $\{z_1, z_2, x_1, \ldots, x_k\}$ and $V(N_2) = \{y_1, y_2\}$.We have $q = k + 4$.

Define a labeling $f:V((P_2 \cup kK_1) + N_2) \rightarrow \{1, 2, \ldots, k + 4\}$ by $f(y_1) = 1$, $f(y_2) = k + 4$, $f(Z_1) = 2$, $f(Z_2) = k + 3$ and $f(x_s) = s + 2$ for $s = 1, 2, \ldots, k$.

It is clear that f induces a consecutive labeling on the edges. Therefore $(P_2 \cup k K_1) + N_2$ is super edge-magic.

Example 2. We give a super edge-magic labeling of $(P_2 \cup 3 K_1) + N_2$

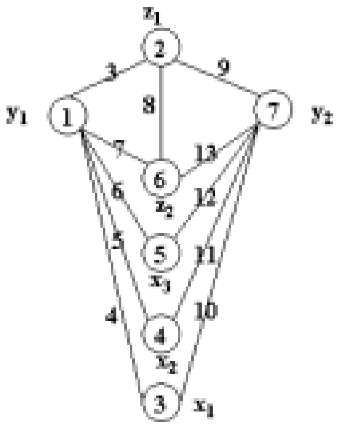


Figure 4.

For any integer $m > 2$ and $n > 1$, the **umbrella graph** $U(m,n)$ is the graph with vertex set $V(U(m, n))= \{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n\}$ and the edge set $E(U(m, n)) = \{(x_i, x_{i+1}):I = 1, 2, \ldots, m - 1\} \cup \{(y_i, y_{i+1}): i = 1, 2, \ldots, n - 1\} \cup \{(x_i, y_1): i = 1, 2, \ldots, m\}$ (see Figure 5).

Theorem 3. For any integer $m > 2$ and $n > 1$, the umbrella graph $U(m, n)$ is super edge-magic.

Proof. The graph $U(m, n)$ has $m + n$ vertices and $2m + n - 2$ edges. Define a labeling $f:V((U(m, n)) \rightarrow \{1, 2, \ldots, m + n\}$ by $f(x_{1+2s}) = s + 1$ for $s = 0, 1, \ldots, \lfloor m/2 \rfloor$.-1.

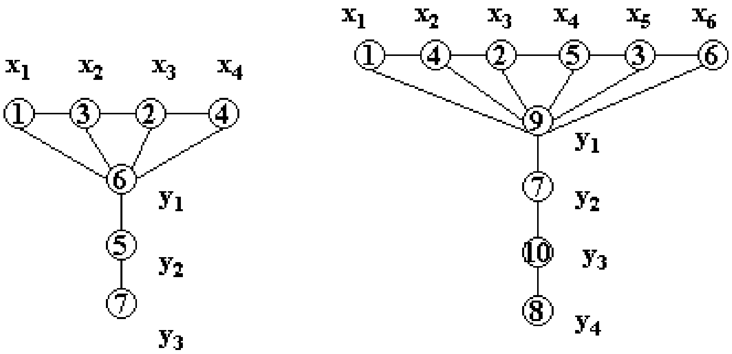
$f(x_{2+2s}) = \lfloor m/2 \rfloor$.+ s + 1 for $s = 0, 1, \ldots, \lfloor m/2 \rfloor$.-2.

$f(y_{1+2t}) = m + \lfloor n/2 \rfloor$.+ 1 + t,.. for $t = 0, 1, \ldots, \lfloor n/2 \rfloor$.-2.

$f(y_{2+2t}) = m + t + 1$, for $t = 0, 1, \ldots, \lfloor n/2 \rfloor$.-2..

It is clear that f induces a consecutive labeling on the edges. Therefore $U(m, n)$ is super edge-magic.

Example 3. In Figure 5 we exhibit a super edge-magic labeling for $U(4, 3)$ and $U(6, 4)$ respectively.



U(4,3) U(6,4)

Figure 5.

We consider a family of graphs which we will call **braid graphs**. For each n > 2, the braid graph B(n) is defined as follows:

$V(B(n)) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ and $E(B(n)) = \{(x_i,x_{i+1}): i = 1, 2, \ldots, n - 1\} \cup \{(y_i, y_{i+1}): i = 1, 2, \ldots, n - 1\} \cup \{(x_i, y_{i+1}): i = 1, 2, \ldots, n - 1\} \cup \{(y_i, y_{i+2}): i = 1, 2, \ldots, n - 2\}$.

Theorem 4. The braid graph B(n) is super edge-magic for all $n \geq 3$.

Proof. The graph B(n) has 2n vertices and 4n-5 edges. We define a vertex labeling f on B(n) as follows:

$f(x_i) = 2i-1$ for $i = 1,\ldots, n$

$f(y_i)= 2i$ for $i = 1,\ldots, n$.

We see that f induces a consecutive labeling on the edge set (see Figure 6).

Example 4.

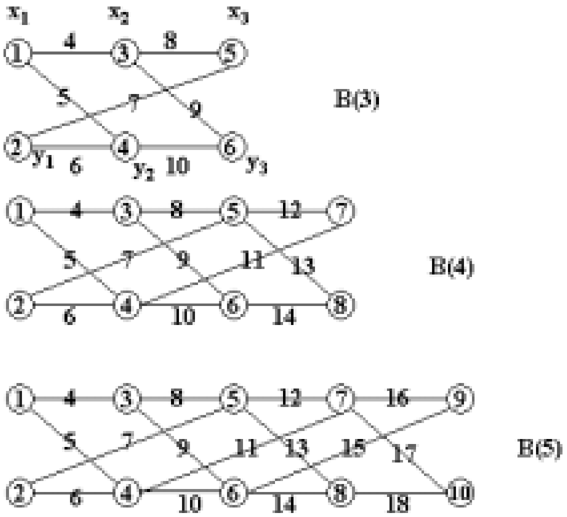


Figure 6.

For integers m, $n \geq 0$, we consider the graph J(m, n) with vertex set $V(J(m, n)) = \{u, v, x, y\} \cup \{x_1, x_2, \ldots, x_m\} \cup \{y_1, y_2, \ldots, y_n\}$ and edge set $E(J(m, n)) = \{(u, x), (u, v), (u, y), (v, x), (v, y)\} \cup \{(x_i, x): i = 1, 2, \ldots, m\} \cup \{(y_i, y): i = 1, 2, \ldots, n\}$.

We will refer to J(m,n) as a Jellyfish graph.

Theorem 5. The Jellyfish graph J(m, n) is super edge-magic for all $m, n \geq 0$.

Proof. We define a labeling $f: V(J(m, n)) \rightarrow \{1, 2, \ldots, m+n+4\}$ as follows:

$f(x_i) = i$ for $i = 1, 2, \ldots, m$

$f(u) = m + 1$, $f(x) = m + 2$, $f(y) = m + 3$, $f(v) = m + 4$,

$f(y_i) = m + 4 + i$ for $i = 1, \ldots, n$.

We see that the labeling f induces a consecutive labeling on E(J(m, n)). Thus, based on the Theorem of Chen, Figueroa et al we conclude that J(m, n) is super edge-magic.

Example 5. A super edge-magic labeling of Jellyfish graph J(3, 4) is shown as follows:

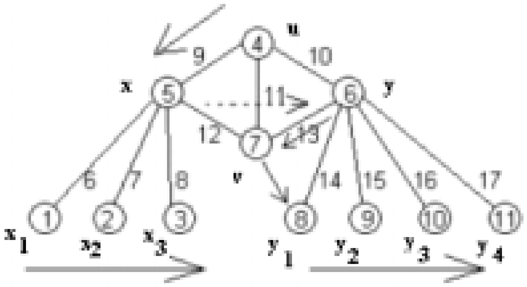


Figure 7.

3. A class of super edge-magic planar graphs.

For n>2 let L_n be the Cartesian product $P_n \times P_2$ of a path on n vertices with a path on two vertices. In 1980 Graham and Sloane proved that L_n is harmonious for n > 2.

Let S = {↑,↓} be the symbol representing the position of the block (Figure 7):

Figure 7.

Let α be a sequence of n symbols of S, i.e. $\alpha \varepsilon S^n$. We will construct a graph by tiling n blocks side by side with their positions indicated by α. We will denote the resulting graph by TB(α) and and refer to it as a **triangular belt.** For simplicity we will denote (↑,↑,↑,…↑) by $\uparrow^n$ and (↓,↓,…↓,↓) by $\downarrow^n$.

Example 6. The triangular belts corresponding to sequences α = (↑, ↓, ↓), β = (↑, ↓, ↓, ↑), γ = (↑, ↓, ↑, ↓), and δ = (↑,↑,↑,↑), respectively are shown in Figure 8.

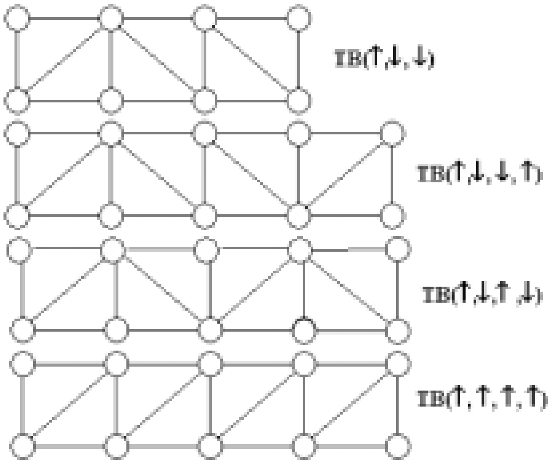


Figure 8.

The following result includes Tsuchiya and Yokomura's result in [31] as a special case.

Theorem 6. For any α in S^n, $n > 1$, the triangular belt TB(α) is super edge-magic.

Proof. The following algorithm indicates a labeling pattern. Label the top vertices of the belts by all odd numbers {1, 3, 5, …, 2n + 1} from left to right successively. Then assign the bottom vertices of the belt from left to right by all the even numbers {2, 4, 6, …, 2n + 2}.

The labeling will provide a super edge-magic labeling.

Figure 9 illustrates the algorithm for TB(α), TB(β), TB(γ), and TB(δ) respectively in Example 6.

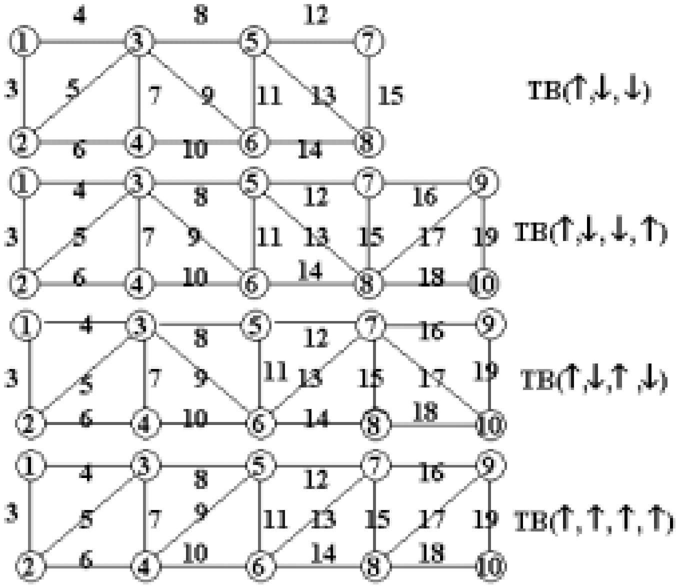


Figure 9.

Corollary 7. For any α in S^n, $n > 1$, the triangular belt TB(α) is sequential and cordial.

In particular, we have the following:

Corollary 8. The 2nd power of P_n is sequential and cordial.

Proof. The 2nd power of P_n is isomorphic to TB(↓, ↓, ..., ↓) or TB(↑, ↑, …, ↑).

The fact that the 2nd power of P_n is sequential was proven by Grace [12] and the fact that it is cordial is proven by Seoud and Abdel Maqsoud [26]. Seoud et al [27] also showed that $P_n^{\ 2}$ is harmonious.

We now consider a class of planar graphs that are formed by amalgamation of triangular belts. For each $n > 1$ and α in S^n we take the triangular belt TB(α) and the triangular belt TB(β), where $k>0$,k blocks with the last block is $\uparrow$.

We rotate TB(β) by 90 degrees counterclockwise and amalgamate the last block with the first block of TB(α) by sharing an edge. The resulting graph is denoted by TBL(n, α, k, β).

The graph TBL(n, α, k, β) has $2(nk + 1)$ vertices and $3(n + k) + 1$ edges with V(TBL(n, α, k, β)) = $\{x_{1,1}, x_{1,2}, \dots, x_{1,n+1}, x_{2,1}, x_{2,2,\dots}, x_{2,n+1}, y_{3,1}, y_{3,2,\dots}, y_{3,k-1}, y_{4,1}, y_{4,2,\dots}, y_{4,k-1}\}$ (see Figure 10 and Figure 11).

__Theorem 9.__ The graph TBL(n, α, k, β) is super edge-magic for all α in S^n and β in S^k with the last block is being $\uparrow$ for all $k > 0$.

Proof. We consider two cases:

Case 1. k = 1. Define $f:V(TBL(n,\alpha,1, \beta))\to\{1,2,\dots 2n+4\}$ as follows:

$f(x_{1,j}) = 4 + 2(j - 1)$ for $j = 1, 2, \dots, n + 1$

$f(x_{2,j}) = 3 + 2(j - 1)$ for $j = 1, 2, \dots, n + 1$

$f(y_{3,1}) = 2$ and $f(y_{4,1}) = 1$.

We observe that f induces a consecutive labeling f^+ on E(TBL($n, \alpha, 1, \uparrow$)).

Case 2. k > 1. Define $f:V(TBL(n, \alpha, k, \beta))\to\{1, 2, \dots,2(nk + 1)\}$ as follows:

$f(x_{1,j}) = 2k + 2j$ for $j = 1, 2, \dots, n + 1$

$f(x_{2,j}) = 2k + 2j - 1$ for $j= 1, 2, \dots, n + 1$

$f(y_{3,j}) = 2j$ for $j= 1, 2, \dots, k$

and $f(y_{4,j}) = 2j - 1$ for $j = 1, 2, \dots, k$..

We observe that f induces a consecutive labeling f^+ on E(TBL(n, α, k, β)).

__Example 7__. Figure 10 illustrate $k = 1$ for Theorem 9.

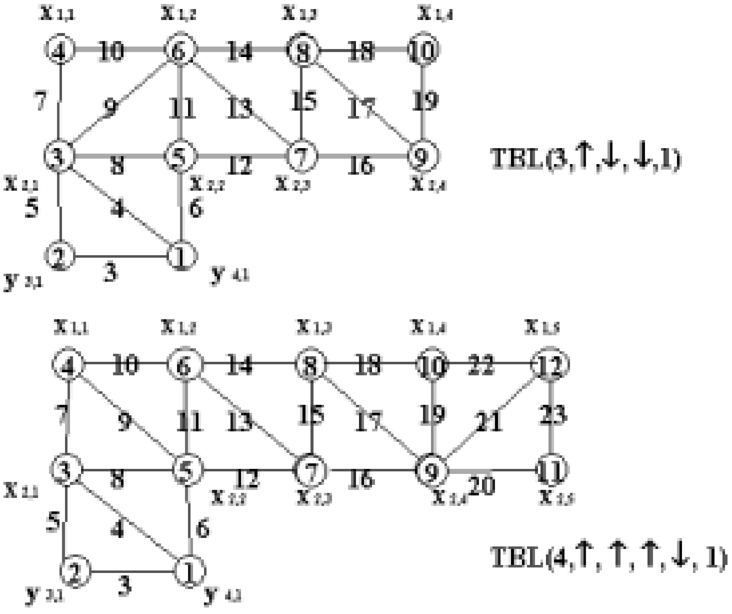


Figure 10.

__Example 8.__ Figure 11 illustrates $k = 2, 3$ for Theorem 9.

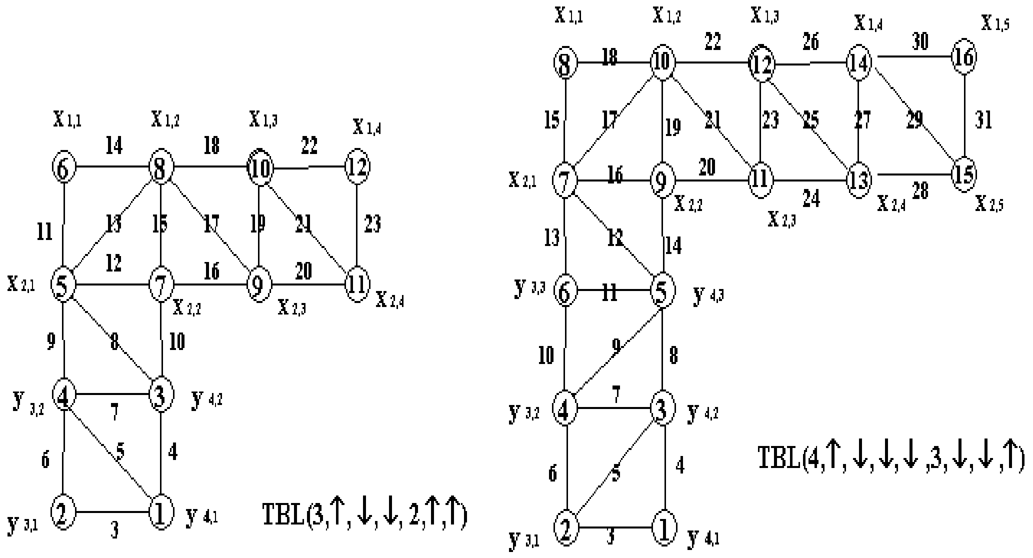


Figure 11.

4. A class of super edge-magic graphs obtained from Mongolian Gers.

For any integers m > 2 and h > 1, the **Mongolian Ger** is the graph M(m, h) with vertex set V(M(m, h)) = {u, $x_{1,1}$, $x_{1,2}$, …, $x_{1,m}$, $x_{2,1}$, $x_{2,1}$, $x_{2,2}$, …, $x_{2,m}$, …, $x_{h,1}$, $x_{h,2}$, …, $x_{h,m}$} and the edge set E(M(m, h)) = {(u, $x_{1,i}$): i = 1, 2, …, m} $\cup$ {($x_{i,j}$, $x_{i,j+1}$): i = 1, 2, …, h, j = 1, 2, …, m} $\cup$ { ($x_{i,j}$, $x_{i+1,j}$): i = 1, 2, …, h-1, j = 1, 2, …, m }. The graph M(m, h) has p = mh + 1 and q = 2mh..

A Mongolian Ger M(4, 3) is shown in Figure 12.

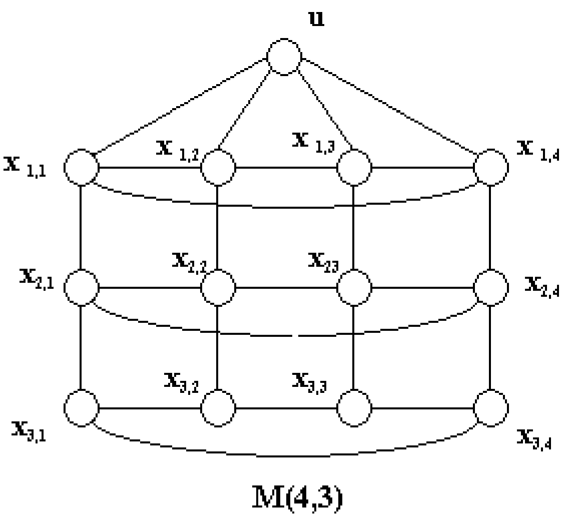


Figure 12.

Consider a subgraph $MT_1(m, h)$ of M(m, h) which is defined as follows:

V ($MT_1(m, h)$) = V (MT (m, h)) and E($MT_1(m, h)$) = E (MT (m, h))\ { (u, $x_{1,2i}$): i = 1, 2, …, $\lfloor m/2 \rfloor$}.

We observe that $MT_1(m, h)$ has 4m - $\lfloor m/2 \rfloor$ edges.

Theorem 10. For each odd $m \geq 3$ and $h \geq 2$, the graph $MT_1(m, h)$ is super edge-magic.

Proof. We label the vertices of $MT_1(m,h)$ from top to bottom and left to right, layer by layer with the numbers $\{1, 2, 3, \ldots, mh + 1\}$ as follows:

Step 1. The top most vertex u is labeled by 1.

Step 2. We label the second layer by the numbers $\{2, 3, \ldots, m + 1\}$as follows: the first vertex $x_{1,1}$ is labeled by 2, from left to right labeled the next vertex with distant two from the previous one consecutively. Since $x_{1,1}, x_{1,2}, \ldots, x_{1,m}$, forms an odd cycle , all the vertices will be completely labeled by the numbers.

Step 3. We label the third layer by the numbers $\{m + 2, m + 3, \ldots, 2m + 1\}$as follows: the second vertex $x_{2,2}$ is labeled by $m + 2$, from left to right labeled the next vertex with distant two from the previous one consecutively. Since $x_{2,1}, x_{2,2}, \ldots, x_{2,m}$, forms an odd cycle, all the vertices will be completely labeled by the numbers.

Inductively, if we already label the k layer

Step k+1.

Case 1. **k is odd.** We label the k + 1 layer by the numbers $\{(k-1)m + 2, (k - 1)m + 3, \ldots, km + 1\}$ as follows:

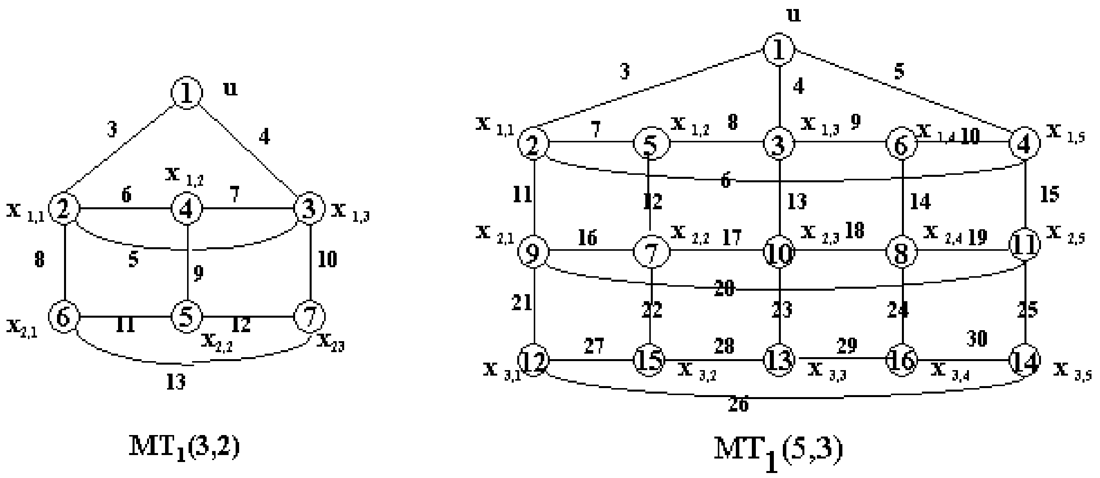
The first vertex $x_{k+1,1}$ is labeled by $(k-1) m + 2$, from left to right labeled the next vertex with distant two from the previous one consecutively. Since $x_{k+1,1}, x_{k+1,2}, \ldots, x_{k+1,m}$, forms an odd cycle, all the vertices will be completely labeled by the numbers.

Case 2. **k is even.** We label the k + 1 layer by the numbers $\{(k-1)m + 2, (k - 1)m + 3, \ldots, km + 1\}$ as follows:

The second vertex $x_{k+1,2}$ is labeled by $(k-1)m + 2$, from left to right labeled the next vertex with distant two from the previous one consecutively. Since $x_{k+1,1}, x_{k+1,2}, \ldots, x_{k+1,m}$, forms an odd cycle, all of the vertices will be completely labeled by the numbers.

We observe that f induces a consecutive labeling f^+ on $E(MT_1(m, h))$, (see Figure 13) .

Example 9.

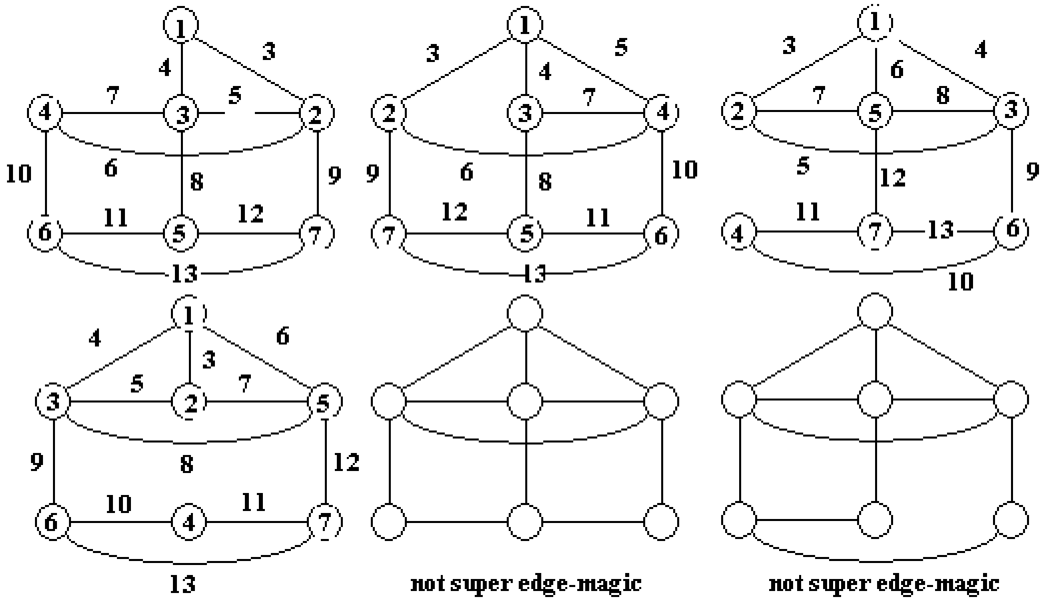


$MT_1(3,2)$ $MT_1(5,3)$

Figure 13.

We see that MT1(m, h) is obtained from M(m, h) by deleting a specific edge from M(m, h). However, not any edge from M(m, h) can be deleted and the resulting graph is super edge-magic.

Example 10. We list here all the possible cases for Mongolian Ger M(3, 2) (Figure 14).



not super edge-magic

not super edge-magic

Figure 14.

Here we propose the following problem:

Question. For any integer m > 2 and h > 1, what edge of M(m, h) can be deleted so that the resulting subgraph is super edge-magic?

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