On Super Edge-magicness of Chain Graphs whose Blocks are Complete Graphs

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ABSTRACT. A (p,q) graph G is *total edge-magic* if there exits a bijective function f: V∪ E → {1.2,…,p+q} such that for each e=(u,v) in E, we have f(u) + f(e) + f(v) is a constant. A total edge-magic graph is called a *super edge-magic* if f(V(G))= {1,2,…,p}. Barrientos [2] defines a chain graph as one with blocks $B_1,B_2,\ldots,B_m$ such that for every i, B_i and B_{i+1} have a common vertex in such a way that the block cut-point graph is a path. In this paper the problem of which chain graphs whose blocks are complete graphs are super edge-magic is studied.

1. Introduction.

In this paper we consider graphs with no loops or multiple edges. For undefined concepts we refer the reader to [4]. A (p,q)-graph G= (V, E) with p vertices and q edges is called ***total edge magic*** if there is a bijectve function f : V∪ E–>{1,2,...,p+q} such that for any (u,v) in E we have a constant s with f(u) + f((u,v)) + f(v)=s. The study of total edge-magic graphs is original initiated by Kotzig and Rosa [11,12]. They called the total edge magic graph as magic graph. Recently, Enomoto et al [7] introduced a particular type of total edge-magic labeling of graphs. A total edge-magic graph is called a ***super edge-magic*** if f(V(G))= {1,2,…,p}. Wallis [33] called super edge-magic as strongly edge-magic. Figure 1 shows a unicyclic graph with 6 vertices with a total edge-magic labeling of sum s=15 and a super edge-magic labeling. with sum s=17.

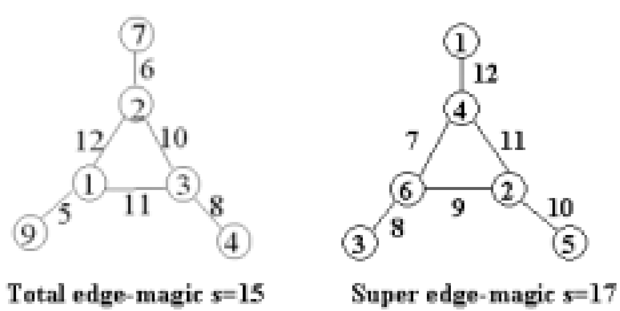

Figure 1.

Kotzig and Rosa [11,12] proved that all cycles, complete bipartite graphs and caterpillars are total edge-magic. A complete graph K_n is total edge-magic if and only if n $\in$ {1,2,3,5,6}. The disconnected graph nK_2 is totally edge-magic if and only if n is odd. In fact, in [11] they showed that every caterpillars and (2k+1) K_2 are super edge-magic. However, the edge-magic labeling of odd cycles given in [12] is not super edge-magic. In [7], Enomoto et al gave a super edge-magic labeling for odd cycles. Godbord and Slater [15] , and independently Craft and Tesar [6] showed that all cycles are total edge-magic.

A subset S of integers is called ***consecutive*** if S consists of consecutive integers. Chen [5] showed that a graph G is super edge-magic if and only if there exists a vertex labeling f such that the two sets f(V(G)) and {f(u)+f(v): (u,v) $\in$ E(G)} are both consecutive. Independently Figueroa-Centeno et al [9] also obtained the same result. They showed that if $f:V(G)\rightarrow\{1,2,\ldots,p\}$ is a bijection of a (p,q)-graph G with S= {f(u)+f(v): uvε E} is consecutive and s=min(S), then f can be extended to a super edge-magic labeling of G by defined f(uv)=p+q+s-f(u)-f(v) for all edge uv of E(G). In light of this result, it suffices to exhibit the vertex labeling of a super edge-magic graph. labeling.(Figure 2). Figure 2 demonstrate how a super-edge magic labeling will arise from a consecutive labeling for the forest St(1) $\cup$St(6).

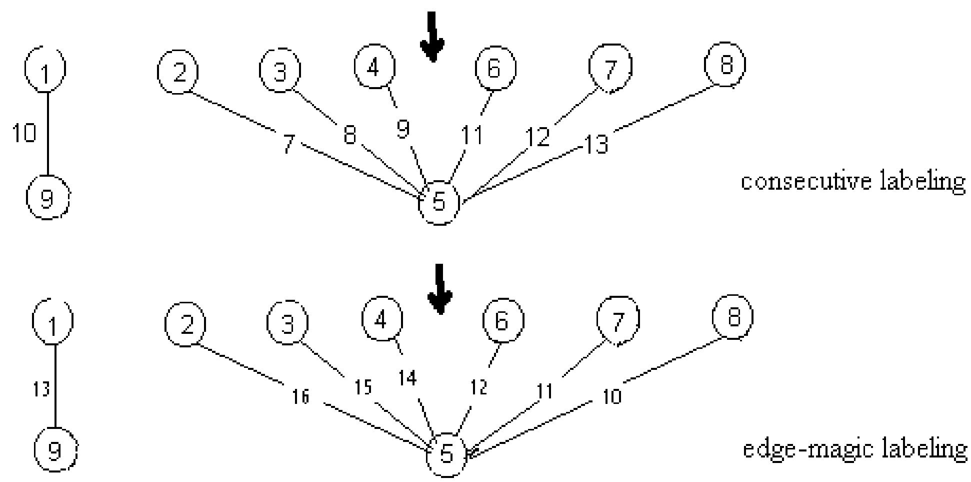

Figure 2

Grace [13] called a (p,q)-graph G **sequential** if there exists a labeling f: $V(G)\rightarrow\{0,1,2,\ldots,q-1\}$ such that the edge labels induced by f(x)+f(y) for each edge (x,y) is c,c+1,c+2,…,c+q-1. Thus by the result of Chen every super edge-magic graph is sequential.

In 1980, Graham and Sloane [14] introduced the concept of **harmonious** graphs. A graph with q edges is harmonious if there exist an injection $f::V(G)\rightarrow\{0,1,2,\ldots,q-1\}$ such that the induced edge labeling f^+: $E(G)\rightarrow$ {0,1,2,…,q-1} defined by f^+({u,v})= f(u)+f(v) (mod q) is injective. Clearly all sequential graphs are harmonious.

Cahit [3] defined a graph cordial if there exists a labeling f: $V(G)\rightarrow Z_2$ with an induced edge labeling f(uv)=f(u)-f(v) (mod 2) such that if $v_f(i)$ and $e_f(i)$ are the number of vertices v and edges e satisfying that f(v)=i and f(e)=i for all iεZ_2, respectively, then $|v_f(0)-v_f(1)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$.

A generalization of harmonious graphs is considered in [23]. The investigation of various graphs which are harmonious or cordial have received significant attention. [, , ,]. Figueroa et al [9] showed that if a (p,q)-graph G with $q \geq p$ is super edge-magic then G is harmonious. Thus the conjecture of Enomoto et al [7] that all trees are super edge-magic is stronger than Graham and Sloane's conjecture that all trees are harmonious.

Barrientos [2] defines a chain graph as one with blocks $B_1, B_2, \ldots, B_m$ such that for every i, B_i and B_{i+1} have a common vertex in such a way that the block cut-point graph is a path.

We will denote the chain graph with n blocks and the sequence of n blocks of complete graphs $\{K(a_1), K(a_2), \ldots, K(a_n)\}$ by $CK(n;(a_1,a_2,\ldots,a_n))$.. We will assume all $a_i \geq 2$. If $a_1=a_2=\ldots=a_n=2$ then $CK(n;(2,2,\ldots,2)) = P_{n+1.}$ It is well known that P_n is super edge-magic. If $a_1=a_2=\ldots=a_n=3$ then $CK(n;(3,3,\ldots,3))$ is the triangular snake which is considered in [23,37]. Moulton [23] showed that a triangular snake is graceful if $n \equiv 1,2 \pmod 4$. Xu [36] showed that a triangular snake is harmonious if and only if $n \neq 3 \pmod 4$.

Recently several classes of graphs had been shown to be total edge-magic ([2 ,4,6,10,13]). In this paper we propose to investigate the existence of super edge-magic labelings for certain classes of chain graphs whose blocks are complete graphs. By means of this investigation we provide new classes of harmonious and cordial graphs.

2. <u>One-point union of two complete graphs.</u>

We will frequently use the following result which is proved by Enomoto et al [9] .

<u>Lemma 1.</u> If a (p,q)-graph G is super edge-magic, then $q \leq 2p-3$.

If n=2, then $CK(2; (a_1,a_2,))$ is the one-point union of two complete graphs. We assume $a_1 \leq a_{2.}$

<u>Theorem 1.</u> For n=2 and a_1= 2, $CK(2; (a_1,a_2,))$ is super edge-magic if and only if $a_2 \leq 4$.

Proof. The super edge-magic labeling of $CK(2; (2,a_2,))$ for a_2 =2,3,4 is depicted as in Figure 3.

By Lemma 1 we see that if a (p,q)-graph G is super edge-magic, then $q \leq 2p-3$. We can see that if $a_2 \geq 5$ then . $q(CK(2; (2,a_2,)))$ is greater than 2 $p((CK(2; (2,a_2,)))$ –3. Hence it is not super edge-magic.

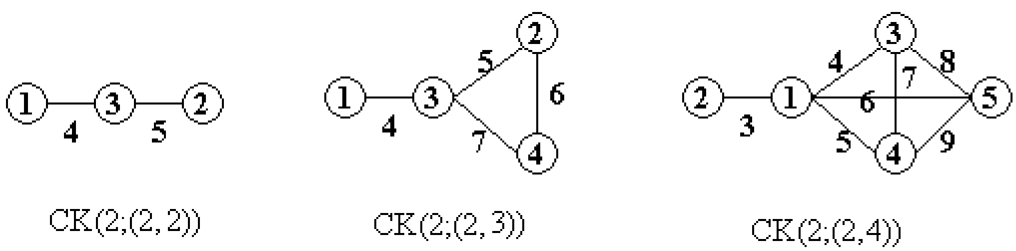

Figure 3.

<u>Theorem 2.</u> For n=2 and a_1= 3, the graph $CK(2;(a_1,a_2,))$ is super edge-magic if and only if a_2=4.

Proof. If a_2=3, then CK(2; (3,3)), is a triangular snake with 2 blocks. If it is super edge-magic then it is cordial. However, this contradicts the results of Cahit [2] that an Eulerian graph whose number of edges is congruent to 2 (mod 4) is not cordial. Therefore CK(2; (3,3)) is not super edge-magic.

The super edge-magic labeling of $CK(2; (3,a_2,))$ for a_2 =4 is depicted as in Figure 4.

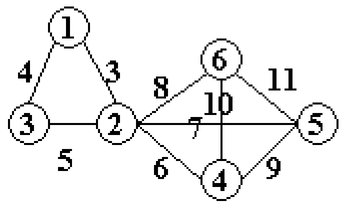

Figure 4.

We can see that if $a_2 \geq 5$ then . q(CK(2; (3,a_2,))) is greater than 2 p((CK(2; (3,a_2,))) –3. Hence by the Lemma 1 it is not super edge-magic.

Using the Lemma 1, we can show that

Theorem 3. For n=2 and $a_1 \geq 4$, CK(2; (a_1,a_2,)),is not super edge-magic for all $a_2 \geq 4$.

3. Chain Graphs with three blocks.

There are several cases to consider for chain graphs with three blocks.. Since the graph CK(3;(a_1,a_2, a_3)). is isomorphic to CK(3;(a_3,a_2, ,a_1)).we will consider one case instead of both.. We will order the sequences by lexicograhical order.

Theorem 4. For n=3, the chain graph CK(3;(a_1,a_2, a_3)). is

(a) super edge-magic if (a_1,a_2, a_3) = (2,2,2),(2,2,4),(2,3,2), (2,3,3), (2,3,4),(2,4,2),(2,4,3),(2,4,4),(3,4,4), (4,2,4), (4,3,4)

(b) not super edge-magic if (a_1,a_2, a_3) = (2,2,3), and (2,2, a_3) $a_3 \geq 5$
(2,3, a_3) $a_3 \geq 5$
(2,4, a_3) $a_3 \geq 5$
(3,2,3), (3,2,4), (3,2, a_3) $a_3 \geq 5$
(3,3, a_3) $a_3 \geq 3$,
(3,4,3) and (3,4, a_3) $a_3 \geq 5$
(4,2, a_3) $a_3 \geq 5$
(4,3, a_3) $a_3 \geq 5$.

Proof. (a) We exhibit a super edge-magic labeling for each graph in Figure 5.

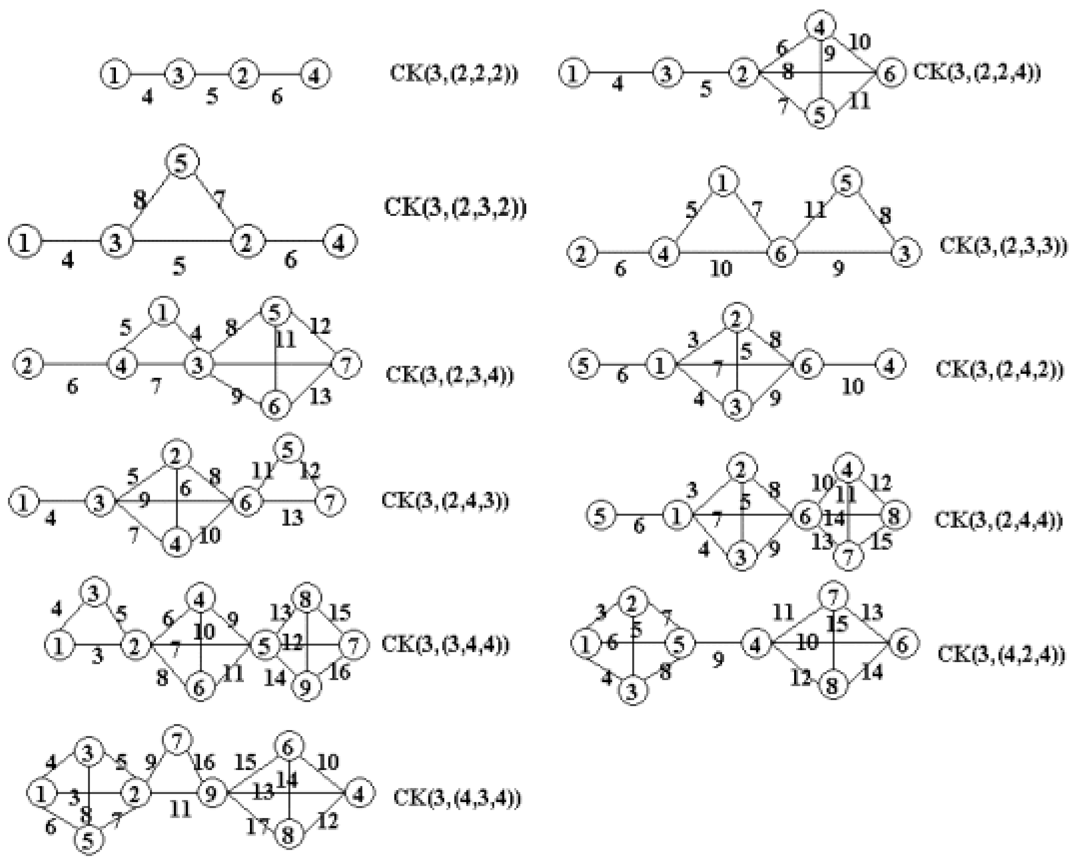

Figure 5.

The following theorem shows that most of the dragon T(n,d) are not super edge-magic.

Theorem 2. If n+d is odd, then T(n,d) is not super edge-magic.
Proof. If T(n,d) is super edge-magic then by the results of Figueroa et al [8], we know it is harmonious. However, this contradicts the results of Liu and Zhang [22] that, when T(n,d) is harmonious, n+d must be even.

4.Chain Graphs with more than three blocks.

Theorem 5. For n=4, the chain graph CK(4;(a_1,a_2, a_3, a_4)). is
(a) super edge-magic if (a_1,a_2, a_3, a_4)= (2,2,2,4),(2,2,3,2),(2,2,3,3),(2,2,4,2),(2,3,2,3),(2,3,2,4), (2,3,3,2),

(2,3,3,3), (2,3,3,4),(2,3,3,5),(2,3,4,2),(2,3,4,4),(3,3,3,3)

(b) not super edge-magic if (a_1,a_2, a_3, a_4)= (2,2,2,3), (3,2,2, 3),(3,2,3,3) and(2,2,2, a_4) $a_4 \geq 5$,

Proof. (a) We exhibit a super edge-magic labeling for each graph in Figure 6a. and Figure 6b.

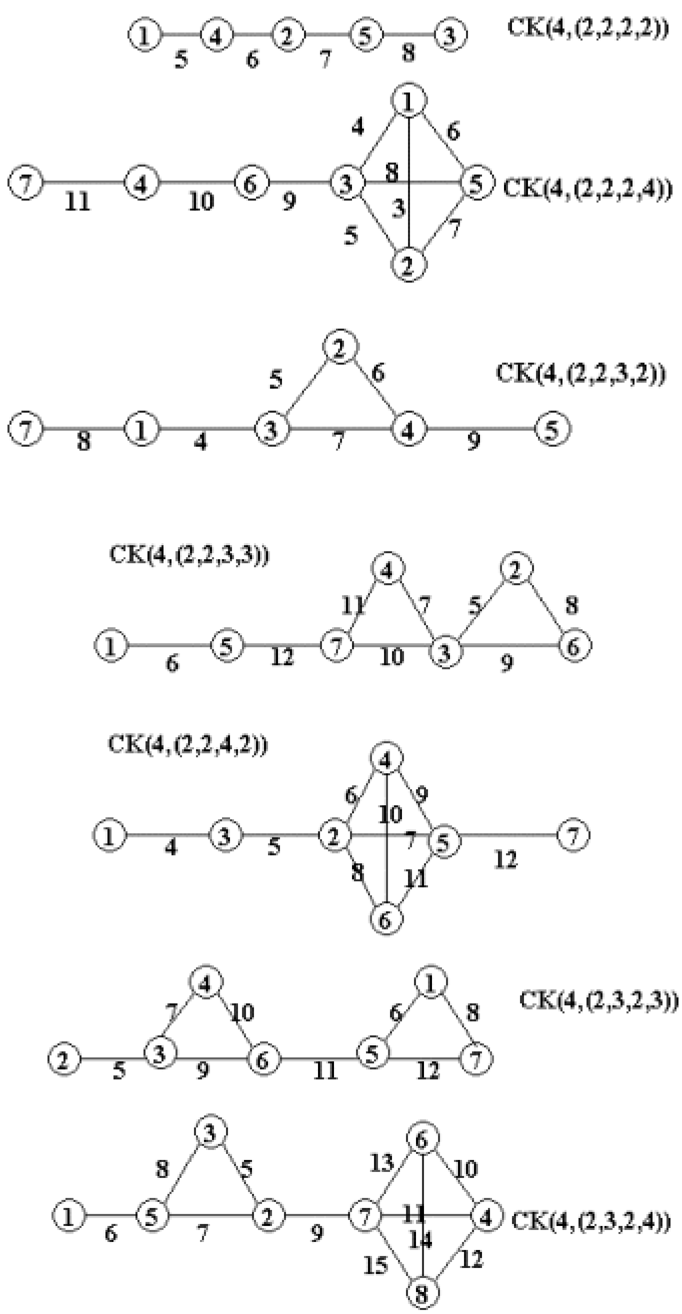

Figure 6a.

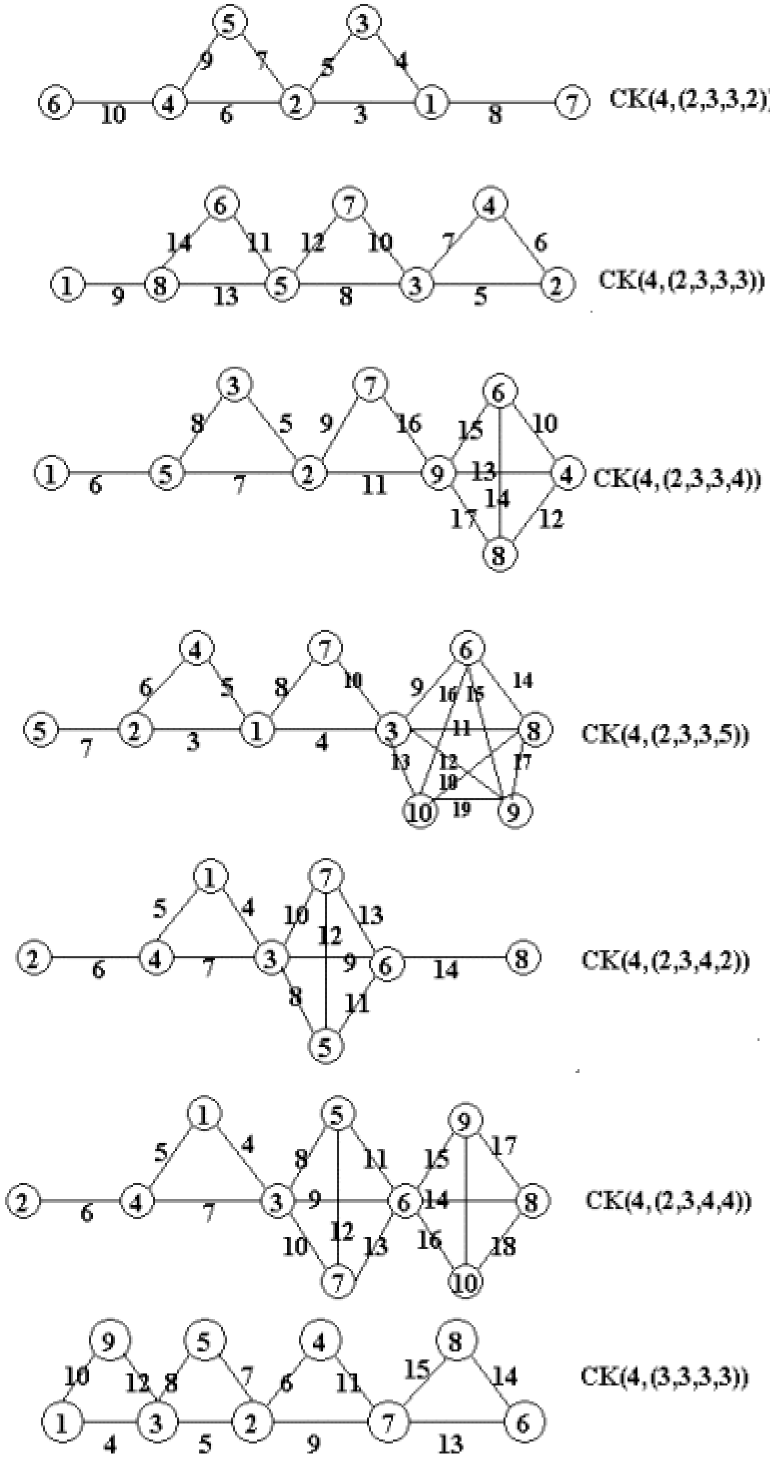

Figure 6 b.

(b)

For the simplicity of presentation, we will present the labeling and the chain graph in the following way: In figure 7, the graph KC(4,(2,3,4,4)) will be presented by (2 4) (4 1 3)(3 5 7 6)(6 9 10 8). The number share in the two neighbour indicate the label of the cutpoint.

Theorem 6. For n $\geq$3, the graphs CK(n, (2,4,4,…4.)) and CK(n, (2,4,4,…4,2.)) are super edge-magic.

Proof. If we include 7, 8 to CK(3,(2,4,2))'s labeling (5,1)(1,2,3,6)(6,4) we obtain a super edge-magic labeling of CK(3,(2,4,4)) i.e. (5,1)(1,2,3,6)(6,4,7,8)

Then we join (7,9) we obtain a super edge-magic labeling of CK(4,(2,4,4,2)), i.e. (5,1)(1,2,3,6)(6,4,8,7)(7,9).

If we add 10,11 to CK(4,(2,4,4,4) i.e. (5,1)(1,2,3,6)(6,4,8,7)(7,9,10,11)

Then we add (10,12) to CK(4,(2,4,4,2)) we obtain a super edge-magic labeling of CK(5,(2,4,4,4,2)) i.e.(5,1)(1,2,3,6)(6,4,8,7)(7,9,11,10)(10,12).

If we add 13,14 we can obtain a super edge-magic labeling of CK(5,(2,4,4,4,4)) i.e. (5,1)(1,2,3,6)(6,4,8,7)(7,9,11,10)(10,12,13,14)

Continue this process, we can have block K_4 as long as we wish. For (n-2,n,n+1,n+2) we have six consecutive numbers: 2n-2,2n-1,2n,2n+1,2n+2,2n+3 so in the initial we have (2,1) to the beginning, then add K_4's to them to get a super edge-magic for each step.

Similarly, we can get a sequence of chain complexes that begin at (1,3,2)

Example 1. The following picture depicits the method of construction of super edge-magic graph CK(n,(2,4,…,2)) and CK(n,(2,4,4,…,4)). (Figure 7)

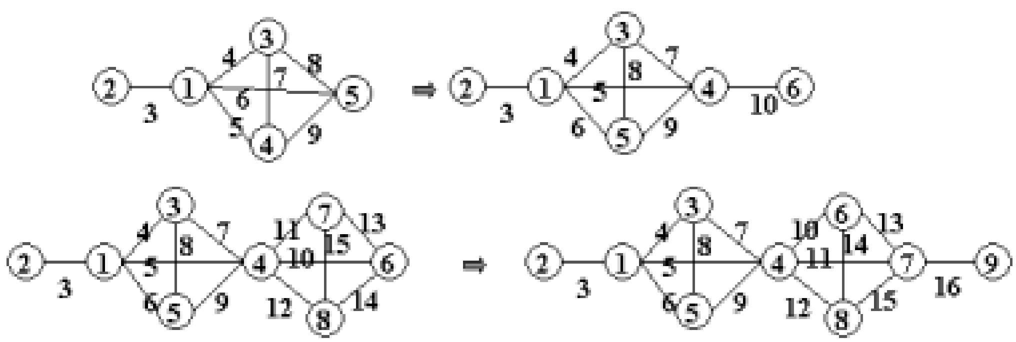

Figure 7.

Theorem 7. For n $\geq$3, the graphs CK(n, (3,4,4,…4.)) and CK(n, (3,4,4,…4,2)) are super edge-magic.

Figure 8 illustrates the method of labeling.

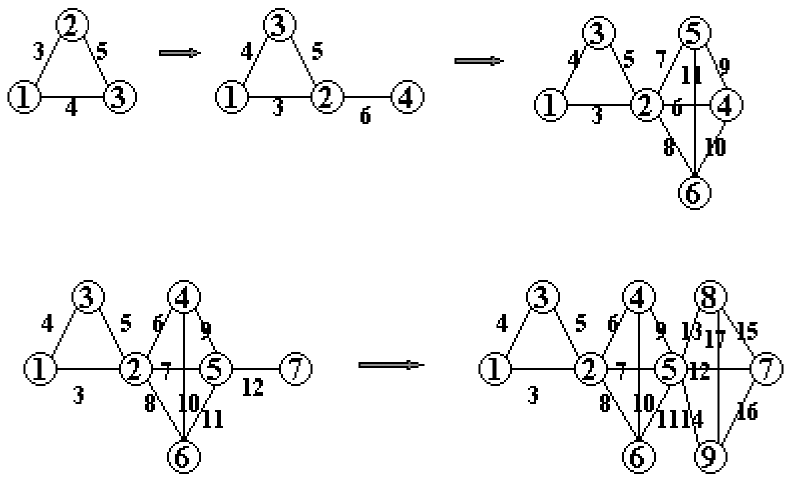

Figure 8

<u>Theorem 8.</u> For n $\geq$3, the graphs CK(n, (2,3,4,4,….)), CK(n, (4,1,4,4,…)) and CK(n, (4,3,4,4,…)) are super edge-magic.

Proof. We label the graphs by the following ways:

CK(n,(2,3,4,4,4....)----- (1,3)(3,4,2)(2,4,6,5)(5,7,9,8)(8,10,12,11).....

CK(n,(4,1,4,4,4....)-----(12,3,5)(5,4)(4,6,8,7)(7,9,11,10)....

CK(n,(4,3,4,4,4....)---- ((1,3,5,2)(2,7,9)(9,4,6,8)(8,10,12,11)....

<u>Theorem 9.</u> For n>4, the graphs CK(n,(2,3,3,5,4,….,4)),and CK(n,(2,3,3,5,4,…,4,2)) are super edge-magic.

Proof. There is a super edge-magic labeling for CK(4,(2,3,3,5)) i.e.(5,2)(2,4,1)(1,7,3)(3,6,8,10,9)

This labeling can be infinitely extended to another super edge-magic labeling for the graphs CK(n, (2,3,3,5,4,….,4)),and CK(n, (2,3,3,5,4,…,4,2)).

<u>Example 2</u>. The chain graph CK(5,(2,3,3,5,4)) has super edge-magic labeling (5,2) (2,4,1) (1,7,3) (3,6,8,10,9) (9,11,13,12)

The chain graph CK(6,(2,3,3,5,4,4)) has super edge-magic labeling (5,2) (2,4,1) (1,7,3) (3,6,8,10,9) (9,11,13,12) (12,14,16,15)

.... etc.

Remark., CK(4,(2,3,3,5)) is equivalent to CK(5,(2,2,2,3,5))

In the following we will write CK(n;($a_1,a_2,\ldots,a_n$)).as CK(n;($a^{[k]}$, $a_{k+1},\ldots,a_n$)). if $a_1=a_2=\ldots=a_k=a$

<u>Theorem 10.</u> The chain graph CK(m+n+1,($4^{[m]}$,2,$4^{[n]}$)) is super edge-magic for all m,n $\geq$0.

Proof. We give a constructive proof. We start at K_2, with label (2,1), then add (0,-1,-2,2) and (1,3,4,5) in both sides. In this new lebeling, (0,-1,-2,2)(2,1)(1,3,4,5), we refill the labeling to the correct form, that is, to start at 1. By adding 3 to each vertex then we get a new labeling, (3,2,1,5)(5,4)(4,6,7,8), which is a super edge-magic labeling of CK(3,(4,2,4)).

Furthermore, we can repeat this step, that is, adding (0,-1,-2,2) and (7,9,10,11) to the both sides of the previous labeling. Refilling the labeling to the correct form, then we have a new super edge-magic labeling of CK(5,(4,4,2,4,4)), that is, (1,2,3,5)(5,4,6,8)(8,7)(7,9,11,10)(10,12,14,13).

With this constrution, we can get a super edge-magic labeling.of the chain graph CK(m+n+1,($4^{[m]}$,2,$4^{[n]}$)) for all m,n $\geq$0.

Example 3. A super edge-magic labeling of CK(3,(4,2,4)).(Figure 9).

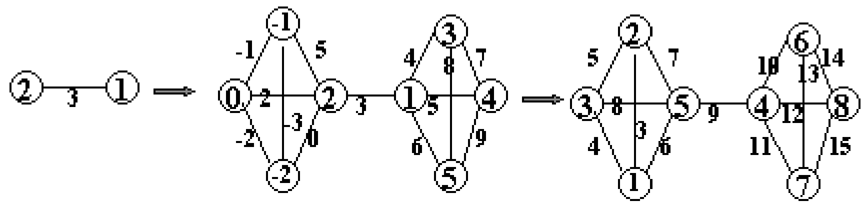

Figure 9.

From super edge-magic labeling of CK(3,(4,2,4)), we can extend to a super edge-magic labeling of CK(5,(4,4,2,4,4)).((Figure 10).

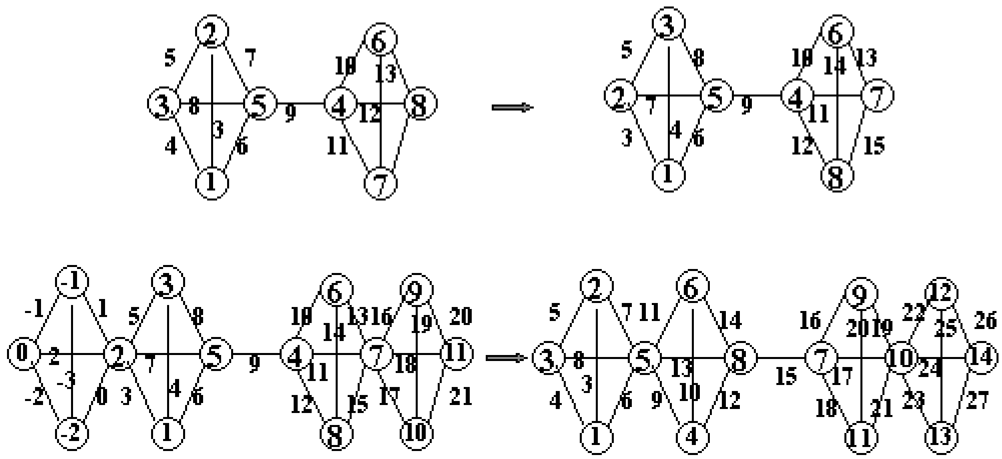

Figure 10.

CK(2,(2,4)),CK(2,(4,2)), CK(3,(4,4,2)), CK(3,(4,2,4)),CK(3,(2,4,4))..........

Of course, in the construction we also have the alternating combination of K_2 and K_4.

Theorem 11. The chain graph CK(m,($3^{[m]}$)) is super edge-magic if and only if m≡0, 1 (mod 4)..

Proof. First of all we will show that CK(m,($3^{[m]}$)) is not super edge-magic if m≡2, 3 (mod 4).

We see that q(CK(m,($3^{[m]}$)))= 3m. If m≡2 (mod 4), then we see that q(CK(m,($3^{[m]}$)))= 2 (mod 4).

As CK(m,($3^{[m]}$)) is an eulerian graph by the result of Cahit it is not cordial and hence it cannot be super edge-magic by the theorem of Figueroa et al [9].

If m≡3 (mod 4) then by the result of Xu [36] it is not harmonious and hence it cannot be super edge-magic by the theorem of Figueroa et al [9].

Now we want to show that CK(m,($3^{[m]}$)) is super edge-magic if m≡0,1 (mod 4)..

Example 4. Two super edge-magic labelings of CK(4,($3^{[4]}$)) (Figure 11)

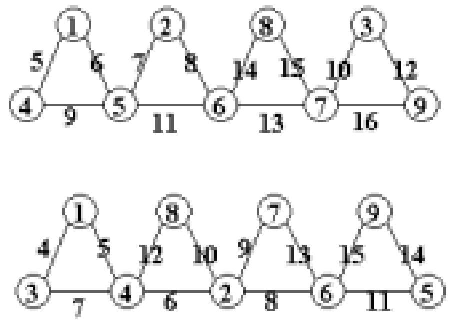

Figure 11.

Example 5. We have a labelling of CK(5,(3,3,3,3,3)), that is,

(1,6,4)(4,2,7)(7,8,10)(10,9,3)(3,5,11).

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