HOMEWORK PROBLEMS ABOUT THE CATENOID

A catenoid is a surface of revolution obtained by rotating a catenary about the Z-axis:

$$\sqrt{X^2 + Y^2} = a \cosh \frac{Z}{a}$$

We must have $a > 0$. This surface can be parametrized by

$$u = \begin{bmatrix}
  a \cosh(Z/a) \cos \theta \\
  a \cosh(Z/a) \sin \theta \\
  Z
\end{bmatrix}$$

The catenoid has a “hole” through which the Z-axis passes. The narrowest part is called the “mouth” of the catenoid. It occurs when $Z = 0$ and the radius of the mouth is $a$.

Consider two circles of the same radius $R$, one in the plane $Z = h$ and one in the plane $Z = -h$, with centers on the Z-axis. Rescaling, we might as well assume $R = 1$. Think of the two circles as made of wire. Dip them in soap film. If the circles are dipped with the Z-axis horizontal, you will get two disks. But if the surfaces are dipped with the Z-axis vertical, you may, under certain conditions, get a portion of a catenoid bounded by those two circles.

Here are the homework problems.

1. Calculate the first and second fundamental forms $g_{ij}$ and $b_{ij}$ of a catenoid.
2. Calculate directly that the mean curvature of a catenoid is zero (it is a minimal surface).

3. If you have two wire circles bounding part of a catenoid, and you pull the circles apart (i.e. increase $h$), eventually the soap film “pops” and you get two disks (or some soap film on the floor). Explain this by showing that there is a maximum $h = h_{\text{max}}$ for which a catenoid can be bounded by those two circles. Does the mouth reach zero diameter when the film pops, or does it pop while the mouth still has a positive diameter? Can you find the numerical value of the $h_{\text{max}}$?

   Hint: We have $a \cosh(h/a) = 1$. Solve this equation for $h$ in terms of $a$.

4. For $h$ smaller than $h_{\text{max}}$, show that there are TWO catenoids bounded by the two circles. The one with smaller $a$ is called the “inner” catenoid and the other is the “outer” catenoid.

5. Calculate the Gauss curvature of a catenoid. This can be used to reason about the stability of these catenoids, but this homework assignment is long enough already, so we will stop with the computation of the curvature.