This paper examines the effect of homogenization on the polynomial degree of refutations in the Polynomial Calculus (PC) and the Hilbert Nullstellensatz (HN) proof systems. It shows that the degree of a homogenized PC refutation is equal to the degree of an HN refutation of the original polynomials and uses this to obtain a $\Omega(n/\log n)$ vs. $O(1)$ degree separation between the HN and PC systems.

The HN and PC systems are studied because of their connection to the open problem of finding lower bounds for Frege systems with modular gates. There are also efficient algorithms for finding PC refutations. Given a set of contradictory equations $Q_1, \ldots, Q_m$ (typically, obtained from a CNF formula) over some field, an *HN refutation* consists in finding polynomials P_i such that $\sum_{i=1}^{m} P_i Q_i = 1$. The *degree* of the refutation is the maximum of the degrees of the $P_i Q_i$'s. Its *size* is $\sum_{i=1}^{m} size(P_i)$ where $size(P_i)$ is the number of monomials in P_i. A *PC refutation* is sequence of polynomials such that each polynomial is either one of the Q_i's or can be obtained from earlier elements of the sequence by linear combination or by multiplication by a term. The last polynomial in a refutation is required to be 1 (Z^j, for some $j > 0$ in the homogenized case). The *degree* of a PC refutation is the maximum degree of a polynomial in the sequence and its *size* is the length of the sequence. To homogenize a set of polynomials, a new variable Z is introduced, and for each polynomial, factors of the form Z^c are multiplied to each term in this polynomial to make all of its terms the same degree.

Using the above notions the paper shows that Z^k is in the ideal generated by a set of homogenized polynomials $Q_1, \ldots Q_m$ iff the original, unhomogenized, polynomials had a degree k Hilbert Nullstellensatz refutation. From this the paper obtains that the degree of a homogenized PC refutation is equal to the degree of an HN refutation of the original polynomials.

The idea of $\Omega(n/\log n)$ vs. $O(1)$ result is to give a lower bound on the degree of homogenized PC refutations and thus using the previously mentioned result get a lower bound HN degree. The paper considers unsatisfiable formulas derived from directed acyclic graphs with one sink. These formulas have one variable for each vertex, the source vertex variables appear as conjuncts in the formula, the sink variable appears negatively as a conjunct, and for each other variable there is a conjunct with the AND of the variable's parents implying the variable. The paper gives an explicit Gröbner basis for the polynomial equations corresponding to this formula as the closure of these equations under an operation for combining two implications. As an application, a result from Buss, et al. [1] that a graph based induction principle requires at least $\Omega(logn)$ is proven using the fact that the basis for this formula will have an element of at least this degree. The paper shows using this explicit basis that if the homogenized equations for a given graph with indegree k have a degree d PC refutation, then the pebbling complexity of the graph is at most $d + k$. The $\Omega(n/\log n)$ lower bound is then a consequence of a known graph of high pebbling complexity.

The paper is very clear and well-organized. The induction principle result was later used in Impagliazzo and Segerlind [2] as a step in their proof of a separation of Frege with counting axioms from Frege with counting gates.

References

[1] S. Buss, R. Impagliazzo, J. Krajíček, P. Pudlák, A. Razborov, and J. Sgall. Proof complexity in algebraic systems and bounded depth frege systems with modular counting. *Computational Complexity* Vol. 6. 1997. pp.256–298.

[2] Russell Impagliazzo and Nathan Segerlind. Counting Axioms Do Not Polynomially Simulate Counting Gates. *Proceedings of the Forty-second Annual Symposium on Foundations of Computer Science*, IEEE Computer Society, 2001. pp. 200–209.