Cook and Reckow [1] showed that if a propositional proof system has polynomial bounded size proofs of every tautology then $\mathsf{NP} = \mathsf{co\text{-}NP}$. The paper under review considers the Frege proof system with the three axioms: (1) Weakening: $a \to (b \to a)$, (2) Distributivity: $(a \to (b \to c)) \to ((a \to b) \to (a \to c))$, and (3) Double Negation: $\neg\neg a \to a$. A proof in this system consists of a finite sequence of formulas $\phi_1, \ldots, \phi_k$ which are either instances of one of these axioms or follow from earlier formulas in the sequence by modus ponens. The length of a proof is the length k of the sequence and the size of the proof is $\sum_{i=1}^{k} |\phi_i|$ where $|\phi_i|$ is the number of symbols in ϕ_i. The paper under review calls the system in which only instances of distributivity and modus ponens are allowed in proofs *distributivity logic*, and calls the system in which only instances of weakening, double negation, and ponens are allowed *weakening and double negation logic*. It is shown that if $\Gamma \vdash \phi$ is provable in distributivity logic then there is a proof of this fact of length $O((|\Gamma| + |\phi|)^3)$ and size $O((|\Gamma| + |\phi|)^4)$. Similarly, it is shown that if $\Gamma \vdash \phi$ is provable in weakening and double negation logic then there is a proof of this fact of length $O(|\Gamma| + |\phi|)$ and size $O((|\Gamma| + |\phi|)^2)$. These results say something about the strength of Frege systems. For instance, if ϕ is a tautology, consider the proof of this in the above Frege proof system. Let Γ be the set instances of weakening and double negation axioms used, then $\Gamma \vdash \phi$ in distributivity logic, and so we get an upper bound how many distributivity axioms were needed. If Γ' is the set of instances of the distributivity axioms used, then as $\Gamma' \vdash \phi$ in weakening and double negation logic, one gets a similar upper bound on how many weakening and double negation axioms are needed. It would be interesting to come up with tautologies for which a matching lower bound to the results in this paper could be obtained.

References

[1] S. Cook and R. A. Reckow. The relative efficiency of propsotional proof systems. Journal of Symbolic Logic. Vol. 44. pp. 36–50. 1979.