One interesting open question in bounded arithmetic is whether it is consistent with S^1_2 that $\mathsf{NP} = \mathsf{co\text{-}NP}$. Krajíček and Pudlák [3] have shown that is consistent with the $\forall\Sigma^{\mathsf{b}}_1$-consequences of S^1_2 that all tautologies have polynomial-sized extended Frege proofs, resolving this question for a fragment of S^1_2. One of the key elements of this result, involving the construction of a chain of cofinal models, lends itself to the intuitionistic bounded arithmetic setting. Using this observation, Buss [1] proved Krajíček and Pudlák's result for the theory IPV^+ which contains IS^1_2, the intuitionistic version of S^1_2. Similarly, Cook and Urquhart [2] consider the extension of PV_1, an equational theory for polynomial time reasoning, to a theory with all finite types, denoted by PV^ω. They prove Krajíček and Pudlák's result for the intuitionistic version of this theory, IPV^ω. In order to try to extend these results and to shed some light on the original problem for S^1_2, the present paper tries to explore how some classically equivalent formulations of S^1_2 behave in the intuitionistic setting. It also gives a new theory for which Krajíček and Pudlák's result holds.

The first result of this paper is that the theory IPV does not prove the double negation of the length minimization principle for NP-formulas, $\neg\neg$LMIN(NP), unless $PV_1 = CPV$. By [4], $PV_1 = CPV$ is known to imply the collapse of the polynomial hierarchy. Here CPV is the conservative expansion of S^1_2 to the language of PV_1 which has symbols for every polynomial time function, and IPV is its intuitionistic variant. The paper then shows that over PV, $\neg\neg$LMIN(NP) intuitionistically proves the PIND(co-NP) axioms. In the classical setting, CPV proves LMIN(NP) and PIND(co-NP) and it is known CPV could be equivalently defined over PV_1 using either scheme. The last result of this paper is that PV+PIND($\mathsf{NP} \cup \mathsf{co\text{-}NP}$) does not intuitionistically prove super-polynomial lower bounds for extended Frege proofs. This theory is not contained in Buss' theory IPV^+ unless the polynomial hierarchy collapses, so this last result represents a new independence result. The proofs of this paper are of a similar flavor to those of the author's earlier papers [5, 6] and are succinct and clear.

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