In classical logic it is known that the theory $S_1^2$ can be axiomatized over the base theory $BASIC$ using either $\Pi_b^1$-$PIND$ or $\Sigma_b^1$-$PIND$ [1]. Here $\Pi_b^1$ formula correspond to $coNP$ predicates and the $\Sigma_b^1$-formulas correspond to the $NP$ predicates. $PIND$ is a “polynomial” induction schema. This paper shows via a model theoretic proof that if one considers intuitionistic theories the analogous result does not hold.

The papers defines a Kripke model to be $T$-normal for some theory $T$ if each of its worlds satisfies $T$. Given two classical models $M$, $N$ of $BASIC$, the model $M$ is said to be a weak end extension of $N$, if $M$ extends $N$ and if its elements which can be bounded by a length of some element in the model extend the corresponding elements in $N$. This notion is meaningful in models of weak arithmetic since in general exponentiation is not total. For the intuitionistic case we say $K'$ is a weak end extension of $K$ if each of its worlds weak end extends a corresponding world in $K$. One of the papers main results is that: Any reversely well-founded $BASIC$-normal weak end extension Kripke model whose terminal worlds model $S_1^2$ forces $BASIC+\Pi_b^{1+}$-$PIND$. The plus in $\Pi_b^{1+}$ is used to denote $\Pi_b^1$-formulas not containing negation or implication. By a result of Johannsen [2] there is a model of $S_2^1$ which has a submodel $M'$ that weak end extends to $M$ such that limited subtraction is not total in $M'$. Since $S_2^1$ can define limited subtraction and is $\forall \Sigma_b^1$-conservative over $IS_2^1$ (intuitionistic $S_2^1$), this shows $BASIC+\Pi_b^{1+}$-$PIND$ does not imply $IS_2^1$.

The second main result of the paper is that the union of the worlds in any linear weak end extension Kripke model of $BASIC+\Pi_b^{1+}$-$PIND$ satisfies $BASIC+\Pi_b^{1+}$-$PIND$. Using this result the paper shows that if $IPV$, the intuitionistic theory of $PV$ plus polynomial induction on $NP$ formulas, proves $coNP$-$PIND$ then the classical closure of of $IPV$, $CPV$, is equal to $PV_1$. By a result of Krajiček, Pušlák, and Takeuti this is known to imply the collapse of the polynomial hierarchy. A corollary of this is that $IS_2^1$ does not prove $\Pi_b^{1+}$-$PIND$ unless the polynomial hierarchy collapses.

The paper is well presented with clear and short proofs.

References
