Resolution is a popular proof system for demonstrating the unsatisfiability of conjunctive normal form (CNF) formulas. In it a CNF formula A is represented as a union (AND) of a set of clauses where each clause represents a disjunct of A and consists of a set of variables and their negations. Resolution has only one rule of inference, from clause $C \cup \{x\}$ and $D \cup \{\bar{x}\}$, where C and D do not involve $x, \bar{x}$, one can derive $C \cup D$. A formula is refuted if one can derive the empty clause using this rules of inference a finite number of times. Resolution is often used as a basis for automated deduction systems. The paper under review shows that the SHORT RESOLUTION REFUTATION problem, that of determining whether a given set of clauses can be refuted within k resolution steps; and the SMALL UNSATISFIABLE SUBSET problem, that of determining whether a given set of clauses contains an unsatisfiable subset of size at most k; are both complete for the parameterized complexity class $W[1]$. The k-step refutability question is of interest because it is closely connected to whether resolution is *automatizable*, that is, whether there is an algorithm that can always find resolution refutations of sets of unsatisfiable clauses in polynomial time in the size of the shortest such refutation. Aleknovich and Razborov [1] had previously shown that resolution is not automatizable unless the class $W[P]$ which contains $W[1]$ is tractable.

The classes $W[1]$ and $W[P]$ can be defined as follows: Given a truth assignment for a set of variables, its weight is the number of variables set to true. The 1-normalised formulas are defined to be the 3CNF formulas. For $t > 1$, the t-normalized formulas are the formulas which have the form an AND of OR's of AND's... with at most t-alternations. The weighted t-satisfiability problem is given an integer k and a t-normalized formula ϕ, does ϕ have a satisfying assignment of weight k? The class of $W[t]$ consists of those problems which fixed parameter time reduce to weighted t-satisfiability; $W[P]$ consists of the those problem which fixed parameter reduce to weighted satisfiability (no restriction on the depth t). Here problem P fixed parameter time reduces to a problem Q if there is algorithm which transforms an instance (x, k) of P into an instance $(x', g(k))$ of Q in time $f(k)|x|^{O(1)}$ where we only require f and g to be computable and that (x, k) is a yes instance of P iff $(x', g(k))$ is a yes instance of Q. A problem P is fixed parameter tractable (FPT) if all instances (x, k) in it can be solved in time $f(k)|x|^{O(1)}$. The hierarchy is $FPT \subseteq W[1] \subseteq W[2] \subseteq \ldots W[t] \subseteq \ldots W[P]$ is conjectured to be strict.

The paper under review gives a fixed parameter time reduction of the

SHORT RESOLUTION REFUTATION and the SMALL UNSATISFIABLE SUBSET to CLIQUE, a well-known $W[1]$-complete problem. Given a graph G, a set of clauses F_G is constructed in time fixed parameter polynomial in the size of G such that G has a clique of size k iff F_G has an unsatisfiable subset F' with at most $k' + 1$ clauses iff F_G has a resolution refutation with at most k' steps. Here $k' = \binom{k}{2} + 2k$. The F_G constructed also shows that several resolution variations such as tree-like resolution, regular resolution, etc. remain $W[1]$-complete. A slight change in the reduction, adding new extension variables, can be used to show both problems remain complete if one restricts oneself to 3CNF formulas. The last section of the paper gives a way to represent a CNF formula by a relational structure. This structure is used as a reduction of SHORT RESOLUTION REFUTATION and the SMALL UNSATISFIABLE SUBSET to MODEL CHECKING another $W[1]$-complete, and thus proves these two problems are contained in $W[1]$. By restricting ones attention to formulas which give rise to structures of locally bounded treewidth, the paper shows the problems becomes fixed parameter tractable since the associated model checking problem is fixed parameter tractable. The tractability algorithm one gets though is not practical and this interesting paper concludes by saying that it might be interesting to come up with practical algorithms for some specific cases such as planar CNF formulas and (k, s)-CNF formulas.

References

[1] M. Alekhnovich, A.A. Razborov. Resolution is not automatizable unless $W[P]$ is tractable. In The 42nd IEEE Symposium on Foundations of Computer Science. FOCS 2001. IEEE Computer Society. 2001. pp. 210–219.