Arithmetics within the Linear Time Hierarchy

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Abstract

We identify fragments of the arithmetic theory S_1, a conservative extension of $I\Delta_0$, that enjoy nice closure properties and have exact characterization of their definable multifunctions. To do this, in the language of S_1, L_1, starting from the bounded formula classes, Σ_i^{b}, which ignore sharply bounded quantifiers when determining quantifier alternations, we define new syntactic classes based on counting bounded existential sharply bounded universal quantifiers blocks. Using these, we define three families of arithmetics: $\breve{S}_1^i$, TLS_1^i and TSC_1^i. $\breve{S}_1^i$ consists of open axioms for the symbols in the language and length induction for one of our new formula classes, $\breve{\Sigma}_{i,1}^{\mathsf{b},\{p(|id|)\}}$. TLS_1^i and TSC_1^i are defined using axioms related to restricted dependent choice sequences for formulas from two other classes within Σ_i^{b} that we define. We prove for $i \geq 1$ that

$$TLS_1^i \subseteq TSC_1^i \subseteq \breve{S}_1^i \preceq_{\forall B(\breve{\Sigma}_{i+1}^{\mathsf{b},\{p(|id|)\}})} TLS_1^{i+1}$$

and that the $\tilde{\Sigma}_i^{\mathsf{b},\{p(|id|)\}}$-definable in TLS_1^i (resp. $\tilde{\Sigma}_i^{\mathsf{b},\{2^{p(||id||)}\}}$-definable in TSC_1^i) multifunctions are L_1-$F\mathsf{L}^{\tilde{\Sigma}_{i,1}^{\mathsf{b}}}[wit]$ (resp. L_1-$F\mathsf{SC}^{\tilde{\Sigma}_{i,1}^{\mathsf{b}}}[wit]$). These multifunction classes are respectively the logspace or SC (poly-time, polylog-space) computable multifunctions whose output is bound by a term in L_1 and that have access to a witness oracle for another restriction on the Σ_i^{b} formulas, $\tilde{\Sigma}_{i,1}^{\mathsf{b}}$. For the $i = 1$ cases, this simplifies respectively to the functions in logspace and SC, Steve's Class, poly-time, polylog-space. We prove independence results related to the Matiyasevich Robinson Davis Putnam Theorem [11] (MRDP) and to whether our theories prove simultaneous nondeterministic polynomial time, sublinear space is equal to co-nondeterministic polynomial time, sublinear space. Specifically, using the notation of this paper, we prove that $TSC_1^1 \nvdash MRDP$ and that $TSC_1^1 \nvdash \mathsf{E}(\mathsf{D}\bar{\mathsf{d}})_{j,1} = \mathsf{U}(\mathsf{D}\bar{\mathsf{d}})_{j,1}$. We show that if $\mathsf{L}^{\tilde{\Sigma}_{i,1}^{\mathsf{b}}} = \mathsf{L}^{\tilde{\Sigma}_{i,2}^{\mathsf{b}}}$ for $i > 0$, then $I\Delta_0 \nvdash MRDP$.

1 Introduction

What makes for a natural decomposition of a theory T into fragments T_i such that $T = \cup_i T_i$? For $PA = \cup_n I\Sigma_n$ and $S_2 = \cup_i S_2^i$, there are natural function classes that are exactly the provably Σ_n (resp. Σ_i^b) definable functions in $I\Sigma_n$ (resp. S_2^i) and these theories prove closure properties for them. $I\Delta_0$, with open axioms for 0, S, $+$, '$\cdot$', and induction for bounded formulas, is well-studied. The Δ_0 formulas express exactly the linear time hierarchy sets, and so $I\Delta_0$ is often the appropriate theory to prove complexity results concerning this hierarchy. Bennett [1] shows $I\Delta_0$ proves its definable functions are closed under recursion on notation if the function being defined is of at most square root growth. Fragments of $I\Delta_0$ have been considered in the context of provability of basic facts from number theory. However, these theories, IE_n, for $n > 0$, where we restrict induction to n bounded quantifier alternations the outermost being existential, do not seem to define natural classes of functions. The main difference between $I\Delta_0$ and S_2 is the additional symbol $x\#y$ with axioms so that it behaves as $2^{|x||y|}$ where $|x|$ is $\lceil \log_2(x+1) \rceil$. In this paper, we try to identify fragments of $I\Delta_0$ that enjoy the same closure properties that the fragments S_2^i of S_2 enjoy and which have exact characterizations of their definable multifunctions.

The theory S_1 is a conservative extension of $I\Delta_0$ to a language L_1 with all the function symbols of S_2 except $x\#y$. Ideally, a decomposition of S_1 into subtheories T_i results in the well-studied theories S_2^i when $\#$ is added back. Write $\Sigma^{\mathsf{b}}_{i,k}$ (resp., $\Pi^{\mathsf{b}}_{i,k}$)for the class of bounded formulas in the language L_k, $k = 1, 2$ that begin with an existential (resp., universal) quantifier block that have at most i quantifier alternations ignoring sharply bounded quantifiers, conjunctions, and disjunctions. S_2^i consists of open axioms for the symbols in L_1, together with $LIND$ (length induction) axioms for $\Sigma^{\mathsf{b}}_{i,k}$ formulas. The presence of $x\#y$ allows the theories S_2^i to express and prove the quantifier exchange property:

$$\forall j < |s| \exists y \leq t A(x, j, y) \Leftrightarrow \exists w \leq bd(t, s) \forall j < |s| A(x, j, \hat{\beta}_{|t|}(j, w))$$

for $A \in \Sigma^{\mathsf{b}}_{i,2}$. Here $\mathrm{bd}(a, b) := 2(2a\#2b)$ and $\hat{\beta}_{|t|}(j, w)$ projects the j block of $|t|$ bits from w. This together with pairing operations means in S_2^i that every $\Sigma^{\mathsf{b}}_{i,2}$-formula is provably equivalent to a formula in prenex normal form, a $\hat{\Sigma}^{\mathsf{b}}_i$ formula, a Σ^{b}_i formula with exactly $i+1$ quantifiers and i quantifier alternations. One can show that S_2^i can in fact be alternatively defined using $LIND$ for $\hat{\Sigma}^{\mathsf{b}}_i$ formulas. If we remove $\#$ from the language, defining S_1^i using $LIND$ for either $\Sigma^{\mathsf{b}}_{i,1}$ or $\hat{\Sigma}^{\mathsf{b}}_{i,1}$ does not seem to yield a theory with a nice characterizations of its definable multifunctions. By Parikh's Theorem, S_1 cannot define $\mathrm{bd}(t, s)$, so this quantifier exchange may not hold in S_1 or its sub-theories, and this can cause problems with the typical arguments which work in the $\#$ setting. It is unclear if a theory with $\Sigma^{\mathsf{b}}_{i,1}$ length induction can prove $\hat{\Sigma}^{\mathsf{b}}_{i,1} = \Sigma^{\mathsf{b}}_{i,1}$. Such a theory's ability to prenexify formulas seems to be at the level of pushing conjunctions and disjunctions into the formula and then using pairing to collapse like bounded existential or universal quantifiers. Doing this, a $\Sigma^{\mathsf{b}}_{i,1}$ formula F could be shown equivalent to a formula F' consisting of a sequence of $(\exists w \leq t)(\forall i < |s|)$ blocks followed by a $\Pi^{\mathsf{b}}_{i-1,1}$ formula. This suggests that to naturally decompose S_1, one might consider theories with schemas involving formulas Ψ, where $\hat{\Sigma}^{\mathsf{b}}_{i,1} \subseteq \Psi \subseteq \Sigma^{\mathsf{b}}_{i,1}$ and in particular look at restrictions of quantifier blocks of the forms $(\exists w \leq t)(\forall i < |s|)$ (Eu quantifier blocks) and $(\forall w \leq t)(\exists i < |s|)$ (Ue quantifier blocks).

Such quantifier blocks naturally arise in the context of Nepomnjaščiĭ's Theorem [12], the proof that for $0 < \epsilon < 1$, $\cup_k \mathsf{NTISP}[n^k, n^\epsilon]$, the languages simultaneous in nondeterministic time n^k for some k and n^ϵ space, is contained in LinH, the linear time hierarchy, as the alternations one gets from the proof are Eu quantifiers. This suggests trying to develop fragments whose definable functions come from a complexity class within $\cup_k \mathsf{NTISP}[n^k, n^\epsilon]$, but with access to an oracle. Previously considered classes within $\cup_k \mathsf{NTISP}[n^k, n^\epsilon]$ are L, logspace, and SC (Steve's Class), that is, $\mathsf{TISP}[\mathsf{poly}, \mathsf{polylog}]$, languages simultaneously in polynomial time and poly-log space. Further, as Krajíček [9] has shown in the $\#$ setting, the $\Delta^b_{i+1,2}$-predicates of S_2^i, those provably equivalent in S_2^i to both a $\Sigma^{\mathsf{b}}_{i,2}$ and $\Pi^{\mathsf{b}}_{i,1}$ formula, are $\mathsf{L}^{\Sigma^{\mathsf{p}}_i}$, it seems especially natural to try to come up with a theory related to L.

Witnessing arguments are a popular sequent calculus proof based approach to showing the definable functions of a theory are contained in some function class. In this approach, handling the sharply bounded ($\forall \leq$:right) case often requires collection-like abilities not generally supported without $\#$. To solve this, we identify syntactic subsets of the Eu quantifier block closure of $B(\hat{\Pi}^{\mathsf{b}}_{i-1})$ (boolean combinations of $\hat{\Pi}^{\mathsf{b}}_{i-1}$ formula), $\mathsf{D\bar{d}H}_i$ and $\mathsf{D\bar{d}H}^{\tau}_i$, which are Δ^b_i in our theories and yet strong enough to capture poly-time, sublinear or $\log \tau$ (for some term $t \in \tau$) space bounded languages via Nepomnjaščiĭ's argument. The name $\mathsf{D\bar{d}H}$ was chosen with the idea that D indicates a bounded quantifier and $\bar{\mathsf{d}}$ indicates a sharply bounded quantifier of the opposite type. As we will see, $\mathsf{D\bar{d}H}$ levels can be defined over $B(\hat{\Pi}^{\mathsf{b}}_{i-1})$ using bounded existential followed by sharply bounded universal quantifiers or over $B(\hat{\Pi}^{\mathsf{b}}_{i-1})$ using bounded universal followed by sharply bounded existential quantifiers. We define classes of formulas, $\tilde{\Sigma}^{\mathsf{b}}_i$ and $\tilde{\Sigma}^{\mathsf{b},\tau}_i$, between Σ^{b}_i and $\hat{\Sigma}^{\mathsf{b}}_i$ which are the bounded existential closures of our $\mathsf{D\bar{d}H}_i$ or $\mathsf{D\bar{d}H}^\tau_i$ formulas. Our witness argument only needs to handle existentials in front of $\mathsf{D\bar{d}H}^\tau_i$ formulas (where τ is $\{|id|\}$ or $\{2^{p(||id||)}\}$) avoiding the need to produce witnesses of polynomial length sequences. Krajíček's result [9], mentioned earlier, that the $\Delta^b_{i+1,2}$-predicates of S^i_2 are $\mathsf{L}^{\Sigma^{\mathsf{p}}_i}$, relies on S^i_2 being able to carry out a maximization argument on the number of 'yes' answered questions of a machine with a Σ^{p}_i oracle. To carry out the analogous argument in the L_1 setting we define classes $\breve{\Sigma}^{\mathsf{b}}_i$ and $\breve{\Sigma}^{\mathsf{b},\tau}_i$ contained in Σ^{b}_i which are the bounded existential closure of a restricted Eu closure of positive query accesses to $\tilde{\Sigma}^{\mathsf{b}}_i$ and $\tilde{\Sigma}^{\mathsf{b},\tau}_i$ formulas.

Given these classes within Σ^{b}_i, we define theories $\breve{S}^i_1$, TLS^i_1, and TSC^i_1, as our proposals of theories which nicely decompose S_1. $\breve{S}^i_1$ is our base theory together with the $\breve{\Sigma}^{\mathsf{b},\{|id|\}}_{i,1}$-$LIND$ axioms, The theories TLS^i_1 are motivated by TLS, an earlier bounded arithmetic for L [6][14] where the weak successive nomination schema they use has been modified into axioms which match the shape of $\mathsf{D\bar{d}H}^\tau_i$ formulas. We show for $i \geq 1$,

$$TLS^i_1 \subseteq TSC^i_1 \subseteq S^i_1 \preceq_{\forall B(\tilde{\Sigma}^{\mathsf{b}}_{i+1})} TLS^{i+1}_1,$$

that the $\tilde{\Delta}^{b,\{|id|\}}_i$ predicates of TLS^i_1 are $\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,1}}$ (for $i = 1$, L), and that the $\tilde{\Delta}^{b,\{2^{p(||id||)}\}}_i$ predicates of TSC^i_1 are $\mathsf{SC}^{\tilde{\Sigma}^{\mathsf{b}}_{i,1}}$ (for $i = 1$, SC). To our knowledge this is the first time that a bounded arithmetic theory whose consequences for some predicate class are SC has been given. In L_2, for $i = 1$, we show $TLS^1_2 \subseteq TLS$ also has L as its $\tilde{\Delta}^{b,\{|id|\}}_i$ predicates and that TSC^1_2 still has SC as its $\tilde{\Delta}^{b,\{2^{p(||id||)}\}}_1$ predicates.

Pollett [14] shows TLS cannot prove $\hat{\Sigma}^{\mathsf{b}}_{1,1} = \hat{\Pi}^{\mathsf{b}}_{1,1}$. We improve this result to TSC^1_1 cannot prove $\mathsf{E}(\mathsf{D\bar{d}})_{j,1} = \mathsf{U}(\mathsf{D\bar{d}})_{j,1}$. Here E and U indicate respectively an existential or a universal bounded quantifier in L_1. This is an improvement as TSC^1_1 can reason about SC, a potentially larger class of languages than L, and $\mathsf{E}(\mathsf{D\bar{d}})_{j,1}$ can express predicates in $\mathsf{E}(\cup_{0<\epsilon<1}\mathsf{TISP}[n^{j\cdot(1-\epsilon)}, n^\epsilon])$ which is likely closer in expressive power to NP than $\hat{\Sigma}^{\mathsf{b}}_{1,1}$ that has roughly the expressive power of NLIN. The proof idea is similar to the earlier result: If TSC^i_1 proves $\mathsf{E}(\mathsf{D\bar{d}})_{j,1} = \mathsf{U}(\mathsf{D\bar{d}})_{j,1}$ then TSC^i_2 proves $\mathsf{SC} = \mathsf{E}(\mathsf{D\bar{d}})_{j,1} = \hat{\Sigma}^{\mathsf{b}}_{j,1} = \hat{\Pi}^{\mathsf{b}}_{j,1}$ and it also collapses the polynomial hierarchy via a padding argument $\mathsf{SC} = \hat{\Pi}^{\mathsf{b}}_{1,2} = \mathsf{U}(\mathsf{D\bar{d}})_{j,2} = \mathsf{E}(\mathsf{D\bar{d}})_{j,2} = \hat{\Sigma}^{\mathsf{b}}_{1,2}$. Together these contradict a No Complementary Speedup Theorem which shows $\hat{\Pi}^{\mathsf{b}}_{j,1} \neq \hat{\Sigma}^{\mathsf{b}}_{j,2}$.

Another application of our results concerns the Matiyasevich Robinson Davis Putnam Theorem [11] (MRDP), the theorem that shows that the Σ_1 and the $\exists_1$ sets are the same and so the Diophantine sets are undecidable. $I\Delta_0$+exp (Gaifman and Dimitracopoulos [7]) is known to prove MRDP. Pollett [14] showed either TLS or S_1 does not prove MRDP. Y. Chen, M Müller, and K. Yokoyama [4] prove if $I\Delta_0$ proves MRDP for small numbers, then $\mathsf{NE} \not\subseteq \mathsf{LinH}$. Using techniques like in our independence results above, we show TSC^1_1 cannot prove MRDP. We also give the conditional result that if $\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,1}} = \mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,2}}$ for some $i > 0$, then $I\Delta_0$ does not prove the MRDP.

The rest of this paper is as follows: Section 2 defines the basic theories, axiom schemas, and so on that we use. Section 3 proves TLS^i_k and TSC^i_k closure properties and proves a lower bound on the multifunction definable in these theories. Section 4 gives a witnessing argument to prove a n upper bound on the multifunctions definable as well as our conservation result. Section 5 proves our MRDP lower bound and independence results.

2 Preliminaries

This section introduces the basic notations needed to express our results and shows that sequence coding is available using terms in the languages that we work with. It then presents our new subclasses of the Σ^{b}_i formulas and uses these to define the subtheories of S_1 we develop in this paper. We express our results in terms of theories and formulas in the language L_1 with non-logical symbols: 0, S, +, '·', $\leq$, $\dot{-}$, $\lfloor \frac{1}{2}x \rfloor$, $|x|$, $\mathrm{PAD}(x,y)$, and $\mathrm{MSP}(x,i)$ or in $L_2 := L_1 \cup \{\#\}$. The symbols 0, $S(x) = x+1$, +, ·, and $\leq$ have the usual meaning. The intended meaning of $x \dot{-} y$ is $\max(x-y, 0)$, $\lfloor \frac{1}{2}x \rfloor$ is x divided by 2 rounded down, and $|x|$ is $\lceil \log_2(x+1) \rceil$. $\mathrm{PAD}(x,y)$, $\mathrm{MSP}(x,i)$, $x\#y$ are intended to mean respectively $x \cdot 2^{|y|}$, $\lfloor x/2^i \rfloor$ and $2^{|x||y|}$. One can generalize L_2 to L_k by defining $\#$ as $\#_2$ and setting for $k > 2$, $L_k = L_{k-1} \cup \{\#_k\}$, where the intended meaning of $\#_k$ is $x\#_k :=^{|x|\#_{k-1}|y|}$. Although our results probably generalize to $k > 2$, for this paper, k as an index is intended to mean $k = 1, 2$.

Many functions and sequence encoding concepts are expressible as L_k-terms. To see this, fix 1 for $S(0)$, 2 for $S(S(0))$, etc. Let $2^{|y|} := \mathrm{PAD}(1, y)$, $2^{\min(|y|,x)} := \mathrm{MSP}(2^{|y|}, |y| \dot{-} x)$, $\mathrm{LSP}(x,i) := x \dot{-} \mathrm{MSP}(x,i) \cdot 2^{\min(|x|,i)}$. Then $\hat{\beta}_{|t|}(x,w) := \mathrm{MSP}(\mathrm{LSP}(w, S(x|t|)), x|t|)$ is the xth block of $|t|$ bits of w and $\mathrm{BIT}(i,x) := \hat{\beta}_1(i,x)$ returns the ith bit of x.

In L_k, we write open_k (or just open if the language is understood) for the class of quantifier-free formulas. open_k formulas reduce to single atomic formulas using terms: $K_\neg(x) := 1 \dot{-} y$, $K_\wedge(x,y) := x \cdot y$, and $K_\leq(x,y) := K_\neg(y \dot{-} x)$ and checking if the term for a given open formula equals 1. New terms can be defined by cases using $cond(x,y,z) := K_\neg(x) \cdot y + K_\neg(K_\neg(x)) \cdot z$. For example, $\max(x,y) := cond(K_\leq(x,y), y, x)$.

Let $B = 2^{|\max(x,y)|+1}$. Thus, B is longer than either x or y. An ordered pair is defined as

$$\langle x, y \rangle := (2^{|\max(x,y)|} + y) \cdot B + (2^{|\max(x,y)|} + x).$$

Its coordinates are $(w)_1 := \hat{\beta}_{\lfloor \frac{1}{2}|w| \rfloor \dot{-} 1}(0, \hat{\beta}_{\lfloor \frac{1}{2}|w| \rfloor}(0,w))$ and $(w)_2 := \hat{\beta}_{\lfloor \frac{1}{2}|w| \rfloor \dot{-} 1}(0, \hat{\beta}_{\lfloor \frac{1}{2}|w| \rfloor}(1,w))$. A number w is a pair if

$$ispair(w) := Bit(w, \lfloor \frac{1}{2}|w| \rfloor \dot{-} 1) = 1 \wedge 2 \cdot |\max((w)_1, (w)_2)| + 2 = |w|$$

holds. For tuples, we write $\langle\langle a_1, a_2, \ldots, a_n \rangle\rangle$ for

$$\langle a_1, \langle a_2, \ldots \langle a_{n-1}, a_n \rangle \ldots \rangle\rangle$$

and define coordinate projection via $\{w\}_j$ via $\{w\}_1 := (w)_1$, $\{w\}_{j+1} := \{(w)_2\}_j$. The usual properties of the terms and formulas above are provable in the theories we will consider in this paper [13].

A quantifier of the form $(\forall x \leq t)$ or $(\exists x \leq t)$ (resp. $(\forall x \leq |t|)$ or $(\exists x \leq |t|)$) where t is a term not containing x is called a *bounded quantifier* (resp. *sharply bounded quantifier*). A formula is *bounded* or Δ_0 (resp. *sharply bounded* or Σ^{b}_0) if all its quantifiers are. For language L, $E_{1,L}$ are those formulas $(\exists x \leq t)\phi$ and $U_{1,L}$ are those formulas $(\forall x \leq t)\phi$ where $\phi \in \mathsf{open}$. $E_{i,L}$ are those formulas $(\exists x \leq t)\phi$ where $\phi \in U_{i-1,L}$ and $U_{i,L}$ are those formulas $(\forall x \leq t)\phi$ where $\phi \in E_{i-1,L}$. We write E_i and U_i when L is understood, and $E_{i,k}$ and $U_{i,k}$ are used for E_{i,L_k} and U_{i,L_k}. For $i > 0$, a $\hat{\Sigma}^{\mathsf{b}}_i$-formula (resp. $\hat{\Pi}^{\mathsf{b}}_i$-formula) is defined to be a E_{i+1}-formula (resp. U_{i+1}-formula) whose innermost quantifier is sharply bounded. To emphasize the language is L_k we write $\hat{\Sigma}^{\mathsf{b}}_{i,k}$ and $\hat{\Pi}^{\mathsf{b}}_{i,k}$. The classes Σ^{b}_i and Π^{b}_i are the closures of $\hat{\Sigma}^{\mathsf{b}}_i$ and $\hat{\Pi}^{\mathsf{b}}_i$ under subformulas, $\wedge$, $\vee$, and sharply bounded quantifications.

The lexicographically Ψ formulas, $\mathsf{L}\Psi$, (for example, $\mathsf{L}\hat{\Sigma}^{\mathsf{b}}_i$) are the formulas that could be made into Ψ formulas by additional quantifications. We write $B(\Psi)$ to denote the class consisting of boolean combinations of Ψ formulas.

As indicated in the introduction, Nepomnjaščiĭ's Theorem [12] connects simultaneously time and space bounded computations to the linear hierarchy, and so to the subtheories of S_1 we will develop. Time-space trade-offs often exploit sublinearly-bounded quantifiers, so it is useful to express terms of growth rate $|x|^\epsilon$ for $0 < \epsilon < 1$. Suppose $\underline{\epsilon}$ has an s bit expansion $0.\epsilon_1 \cdots \epsilon_s$. The first s-bits of $\underline{\epsilon} \cdot x$ are $(\underline{\epsilon} \cdot x)_s := \sum_{i=1}^{s} \mathrm{MSP}(x, \epsilon_i \cdot i)$. Define $\ell^\epsilon_s(x)$ as $MSP(2^{||x||}, ((1-\underline{\epsilon})||x||)_s)$. This approximates $2^{\underline{\epsilon}||x||}$, and hence, the growth of $|x|^{\underline{\epsilon}}$. For this paper, we use fixed $\underline{\epsilon}$, so we drop the underline underneath ϵ, as ϵ will not be used as a variable. To further simplify notation without affecting our results, we assume that ϵ has a finite binary expansion, and that we choose s to be the length of this expansion, so we write ℓ^ϵ for ℓ^ϵ_s.

We now develop time-space tradeoff formula hierarchies. Given a term t, define a nondecreasing term $t^\star$ with $t \leq t^\star$ by recursively replacing subterms of t of the form $\mathrm{MSP}(s, s')$ or $s \dot{-} s'$ with just s. In bounded arithmetic, computations are usually expressed using sequences. Let $\ell \in L_k$ be a unary term such that $|\ell(z)| \leq m \cdot \ell^{1-\epsilon}(h(z))$ for some L_k term h and $m \in \mathbb{N}$. Such an ℓ can be used to bound the space that is used in a single configuration of a computation. If $1 \geq \epsilon > 0$, then this configuration would be of sublinear space. A sequence of t such bounded configurations can be viewed as defining a t time computation, under the assumption that we have appropriately defined what it means for one configuration to follow another. To that end a formula $F_{\vec{t},\vec{B}}(C, C', \vec{a})$ is **(ℓ, ϵ) steppable** if it is of the form

$$\bigvee\!\!\!\!\bigvee_{i \leq n} [\min(t_i(C, \vec{a}), \ell(\max_{i \leq n}(t_i^\star))) = C' \wedge B_i(C, C', \vec{a}) \wedge$$
$$\bigwedge\!\!\!\!\bigwedge_{j<i} (\min(t_j(C, \vec{a}), \ell(\max_{i \leq n}(t_i^\star)) \neq C' \vee \neg B_j(C, C', \vec{a}))],$$

for some formulas B_i, terms t_i, and where we require $B_n := C = C$. Here $\max_{i \leq n}(x_i)$ is inductively defined as $\max_{i \leq 2}(x_i) = \max(x_1, x_2)$, $\max_{i \leq n}(x_i) := \max(\max_{i \leq n-1}(x_i), x_n)$. The condition on B_n ensures at least one of the $\vee$ clauses holds. The conjunctive clauses ensure that C' has value t_i of the least i such that B_i holds. Since the whole formula is an OR of finitely many ANDs, even in the base open theories we will soon define, we can finitistically reason that there is a unique C following a given C. So

$$\forall C \exists! C' \leq \ell(\max_{i \leq n}(x_i)) F_{\vec{t},\vec{B}}(C, C', \vec{a})$$

will be provable in the weakest theories we later define. As hopefully the introduction at the start of this section on sequence coding and pairing using just terms suggests, quite general single step computations can be represented using steppable formulas, in particular, they can be used to represent single steps, from one configuration to the next, of space bounded Turing Machines. Our next notion is used to model several steps of a space bounded computation. We assume now $|\ell(z)| \leq m \cdot \ell^{1-\epsilon}(t_1(z))$ for some L_k term t_1 and $m \in \mathbb{N}$ we use in the following definition. We say $F_{\vec{t},\vec{B}}$ is **(ℓ, ϵ) iterable** if it is **(ℓ, ϵ) steppable** or if it is an **(ℓ, ϵ) iteration** formula, $\mathrm{Iter}_{t_1,t_2,B_1}(C, C', c, \vec{a})$:

$$(\exists w \leq 2^{2 \cdot m |t_1^\star|})(\forall u \leq \ell^\epsilon(t_1^\star))[C \leq \ell(t_1) \wedge \hat{\beta}_{|\ell(t_1^\star)|}(0, w) = C \wedge$$
$$C' \leq \ell(t_1) \wedge t_2(\hat{\beta}_{|\ell(t_1^\star)|}(\min(\ell^\epsilon(t_1^\star), S(c)), w)) = C' \wedge$$
$$B_1(\hat{\beta}_{|\ell(t_1^\star)|}(\min(u, c), w), \hat{\beta}_{|\ell(t_1^\star)|}(S(\min(u, c)), w), \vec{a})]$$

where B_1 is (ℓ, ϵ) iterable. Given $|\ell(z)| \leq m \cdot \ell^{1-\epsilon}(t_1(z))$, a sequence $\ell^\epsilon(t_1^\star)$ blocks of $|\ell(t_1^\star)|$ bits can be represented by a $w \leq 2^{2 \cdot m |t_1^\star|}$. The min expressions and c are to facilitate our proofs, allowing for sequences of fewer than ℓ^ϵ values. To get ℓ^ϵ values, set $c = \ell^\epsilon(s)$. If $F_{\vec{t},\vec{B}}$ is (ℓ, ϵ) iterable, we call a formula $G_{\vec{t},t',\vec{B}}$ of the form

$$\exists C' \leq \ell(t)[F_{\vec{t},\vec{B}}(C, C', \ell^\epsilon(t_1^\star), \vec{a}) \wedge t'(C') = 1]$$

for some term t', **(ℓ, ϵ) iterable with accept state**. If t' is the term 1, we call the accept state **trivial**.

Remark 1 *Notice if a formula ϕ is (ℓ, ϵ) iterable with accept state then $\neg\phi$ is equivalent to the same formula but with clause $1 \dot{-} t'(C) = 1$.*

Let Q indicate one of E (bounded existential), e (sharply bounded existential), U (bounded universal), or u (sharply bounded universal). We write $Q\Psi$ for formulas with a Q quantifier followed by a formula in Ψ. Given that any L_1 term $t(x)$ has growth rate bounded by $2^{k|x|}$ for some fixed $k \in \mathbb{N}$, iterable formulas in the language in L_1 can express sequences of configurations of at most length ℓ^ϵ if each of the individual configurations is of length bounded by $O(\ell^{1-\epsilon})$. To express polynomial computations of space bounded configurations, we can take iterations of previously defined iterable functions. Each such iteration can be viewed as adding either an Eu quantifier block, or in light of Remark 1, an Ue quantifier block. If we let D represent either an E or a U bounded quantifier and $\overline{\mathsf{d}}$ represent a sharply bounded quantifier of the opposite kind, one can view applying finitely many such iteration operations as creating a formula in a hierarchy of formulas based on $\mathsf{D}\overline{\mathsf{d}}$ quantifiers. To be more precise, we define:

$$
\begin{aligned}
(\mathsf{D}\overline{\mathsf{d}})_0(\Psi) &:= \{\phi\,|\phi \in \Psi\}\\
(\mathsf{D}\overline{\mathsf{d}})_{m+1}(\Psi) &:= (\mathsf{D}\overline{\mathsf{d}})_m(\Psi) \cup \{\phi \mid \phi \text{ is a substitution instance of an } (\ell,\epsilon) \text{ iterable formula } F_{\vec{t},\vec{B}}\\
&\qquad \text{or one with accept state } G_{\vec{t},t',\vec{B}}, \text{ where for each } i,\ B_i \in (\mathsf{D}\overline{\mathsf{d}})_m(\Psi)\}\\
(\mathsf{D}\overline{\mathsf{d}})^{\tau}_m(\Psi) &:= \{\phi|\phi \in (\mathsf{D}\overline{\mathsf{d}})_m \text{ and } \ell \le \ell' \in \tau \text{ for all } \ell \text{ used in iterations in } \phi\}\\
\mathsf{D}\overline{\mathsf{d}}\mathsf{H}(\Psi) &:= \{\phi|\phi \in \cup_m(\mathsf{D}\overline{\mathsf{d}})_m(\Psi) \text{ is with accept state}\}\\
\mathsf{D}\overline{\mathsf{d}}\mathsf{H}^{\tau}(\Psi) &:= \{\phi|\phi \in \mathsf{D}\overline{\mathsf{d}}\mathsf{H}(\Psi) \text{ and } \ell \le \ell' \in \tau \text{ for all } \ell \text{ used in iterations in } \phi\}
\end{aligned}
$$

We write $(\mathsf{D}\overline{\mathsf{d}})_m$ for $(\mathsf{D}\overline{\mathsf{d}})_m(\hat{\Sigma}^{\mathsf{b}}_0)$, $\mathsf{D}\overline{\mathsf{d}}$ for $\mathsf{D}\overline{\mathsf{d}}\mathsf{H}(\hat{\Sigma}^{\mathsf{b}}_0)$, etc. By default in this case we will assume the language is L_1, however, if we want to emphasize the language is L_k, $k = 1, 2$, we will write $(\mathsf{D}\overline{\mathsf{d}})_{m,k}$, etc.

The following result is essentially Nepomnjaščiĭ's Theorem [12], but expressed in terms of the hierarchies we have just introduced.

Lemma 1 *For* $0 < \epsilon < 1$, $\mathsf{TISP}[n^{k\cdot\epsilon}, n^{1-\epsilon}] \subseteq (\mathsf{D}\overline{\mathsf{d}})_{k+1}$. *As a consequence,* L *and* SC *are contained in* $\mathsf{D}\overline{\mathsf{d}}\mathsf{H}$.

Proof. Fix an $0 < \epsilon < 1$. Without loss of generality we can assume ϵ has a finite binary expansion, as if not, we could choose ϵ' with a finite binary expansion such that $\epsilon < \epsilon' < 1$ and observe $\mathsf{TISP}[n^{k\cdot\epsilon}, n^{1-\epsilon}] \subseteq \mathsf{TISP}[n^{k\cdot\epsilon'}, n^{1-\epsilon'}]$. Given this assumption on ϵ, as $n = |x|$ is the length of an instance x, we note that

$$\mathsf{TISP}[n^{k\cdot\epsilon}, n^{1-\epsilon}] = \mathsf{TISP}[|x|^{k\cdot\epsilon}, |x|^{1-\epsilon}] \subseteq \mathsf{TISP}[(\ell^{\epsilon}(x))^k, \ell^{1-\epsilon}(x)],$$

so it suffices to show $\mathsf{TISP}[(\ell^{\epsilon}(x))^k, \ell^{1-\epsilon}(x)] \subseteq (\mathsf{D}\overline{\mathsf{d}})_{k+1}$. Let M be an m tape Turing Machine with alphabet K and states Q. Since Q and K are finite, each symbol and each state can be given a code as a binary string of some finite length v. Natural numbers of length $v \cdot t + 1$, can be used to represent, ignoring the most significant bit, t tape squares of a tape of M. A configuration of M could be represented by a tuple:

$$\langle\langle q, \text{ltape}_1, \text{rtape}_1, \ldots, \text{ltape}_m, \text{rtape}_m\rangle\rangle$$

, where q is the code for the current state, ltape_i represents as the tapes squares from the start of tape i up to and including the tape head, and rtape_i represents affected tapes squares to the right of tape head. Without loss of generality, we use the number 1 for the configuration that follows any configuration involving an accept state. We also have that the configuration that follows 1 is also 1. To show $\mathsf{TISP}[(\ell^{\epsilon}(x))^k, \ell^{1-\epsilon}(x)] \subseteq (\mathsf{D}\overline{\mathsf{d}})_{k+1}$, we prove by induction on k that there is a $(2^{\ell^{1-\epsilon}}, \epsilon)$ iterable formula $F_{\vec{t},\vec{B}}(C, C')$ in $(\mathsf{D}\overline{\mathsf{d}})_{k+1}$ that, given C, holds for the exactly one C' representing the configuration of M after $(\ell^{\epsilon}(x))^k$ steps each of which is $\ell^{1-\epsilon}(x)$ space bounded. The result then follows for each particular k by substituting a term that computes the start configuration from x for C and by substituting 1 for C'.

When $k = 0$, since M is deterministic, given the symbols being read by each tape head and a state q, there is a unique available transition to some new state and tape updates. Given this and the ability to project blocks of bits from numbers using the $\hat{\beta}_v$ term, we can define terms $t_\tau(C)$ and formula open formula $B_\tau(C)$ for each transition τ of m's transition function, where $B_\tau(C)$ checks if the configuration encoded by C satisfies the conditions to apply the transition τ and $t_\tau(C)$ manipulate C to make a next configuration C'. Then taking these B_τ's as our $\vec{B}$, t_τ's as our $\vec{t}$, we can make an $(2^{\ell^{1-\epsilon}}, \epsilon)$ steppable formula $F_{\vec{t},\vec{B}}(C, C')$ such that $\forall C \exists ! C' \leq \ell(t) F_{\vec{t},\vec{B}}(C, C', \vec{a})$ and the satisfying C' for a given C is the next configuration following C that M would compute.

For the induction step, suppose we have a $(2^{\ell^{1-\epsilon}}, \epsilon)$ iterable formula $F_{\vec{t},\vec{B}}(C, C')$, that, given C, holds for exactly one C' representing the configuration of M after $(\ell^{\epsilon}(x))^k$ steps each of which is $\ell^{1-\epsilon}(x)$ space bounded. Let $B'_1 := F_{\vec{t},\vec{B}}(C, C', c, \vec{a})$. If $F_{\vec{t},\vec{B}}(C, C')$ is a steppable formula take $t'_1 := \max_{i \leq n}(t^{\star}_i)$ and $t'_2(x) := x$. If $F_{\vec{t},\vec{B}}(C, C')$ is an $(2^{\ell^{1-\epsilon}}, \epsilon)$ iteration formula, $\mathrm{Iter}_{t_1,t_2,B_1}(C, C', c, \vec{a})$, take $B'_1 := F_{\vec{t},\vec{B}}$, take $t'_1 = t_1$ and $t'_2 = t_2$. Given our base case, $t'_2(x) = x$, so we are not really using t_2 in this proof, we will use it when we give a definition of $Numones$ later in the paper. Consider for both cases the formula $F_{\vec{t'},\vec{B'}}(C, C', c, \vec{a}) = \mathrm{Iter}_{t'_1,t'_2,B'_1}(C, C', c, \vec{a})$. So $F_{\vec{t'},\vec{B'}}(C, C', c, \vec{a})$ is:

$$\begin{aligned}(\exists w \leq 2^{2\cdot m|t'^{\star}_1|})(\forall u \leq \ell^{\epsilon}(t'^{\star}_1))[&C \leq 2^{\ell^{1-\epsilon}(t'_1)} \wedge \hat{\beta}_{\ell^{1-\epsilon}(t'^{\star}_1)}(0, w) = C \wedge \\ &C' \leq 2^{\ell^{1-\epsilon}(t'_1)} \wedge \hat{\beta}_{\ell^{1-\epsilon}(t'^{\star}_1)}(\min(\ell^{\epsilon}(t'^{\star}_1), S(c)), w) = C' \wedge \\ &F_{\vec{t},\vec{B}}(\hat{\beta}_{\ell^{1-\epsilon}(t'^{\star}_1)}(\min(u, c), w), \hat{\beta}_{\ell^{1-\epsilon}(t'^{\star}_1)}(S(\min(u, c)), w), \ell^{\epsilon}(t'^{\star}_1), \vec{a})].\end{aligned}$$

So a w satisfying the outer existential must consist of a sequence of $\ell^{\epsilon}(t'^{\star}_1)$ configurations, each of at most $\ell^{1-\epsilon}(t'^{\star}_1)$ bits, starting at C and ending at C' such that between consecutive configurations there exists a computation of $(\ell^{\epsilon}(x))^k$ steps each of which is $\ell^{1-\epsilon}(x)$ space bounded. Since these intermediate computations are unique, for a given C, this formula holds for exactly one C' representing the configuration of M after $(\ell^{\epsilon}(x))^{k+1}$ steps each of which is $\ell^{1-\epsilon}(x)$ space bounded. This show the induction step and completes the proof. □

We now define a second Eu quantifier hierarchy. This hierarchy is motivated by Krajíček's result [9] that the $\Delta^b_{i+1,2}$-predicates of S^i_2 are $\mathsf{L}^{\Sigma^{\mathsf{p}}_i}$ which relies on S^i_2 being able to carry out a maximization argument on the number of 'yes' answered questions of a machine with a Σ^{p}_i oracle. To do this we want to modify our notion of steppable to allow for steps that might involve a query to an oracle formula such that if the answer is positive then the answer must be correct. Suppose $F_{\vec{t},\vec{B}}$ is a steppable formula where $\vec{t}$ are terms and $\vec{B}$ are open formulas. Let $\mathrm{Check}(C')$ be an open formula, and $\mathrm{Query}(C)$ be a term. A **(ℓ, ϵ) query steppable** formula with $A \in \Psi$ is a formula of the form:

$$F_{\vec{t},\vec{B}}(C, C', c, \vec{a}) \wedge (\mathrm{Check}(C') \supset A(\mathrm{Query}(C), \vec{a})).$$

Define $\widetilde{\mathsf{EuH}}(\Psi)$ and $\widetilde{\mathsf{EuH}}^{\tau}(\Psi)$ in the same way as $\mathsf{D\bar{d}H}(\Psi)$ and $\mathsf{D\bar{d}H}^{\tau}(\Psi)$, but using (ℓ, ϵ) query steppable rather than (ℓ, ϵ) steppable in the definitions. Define $\widetilde{\mathsf{UeH}}(\Psi)$, to be the class of formulas whose negations are logically equivalent to $\widetilde{\mathsf{EuH}}(\Psi)$ formulas.

We define our variations on the $\Sigma^{\mathsf{b}}_{i,k}$ and $\hat{\Pi}^{\mathsf{b}}_{i,k}$ formulas:

$$\begin{aligned}
\breve{\Sigma}^{\mathsf{b}}_{0,k} &= \tilde{\Sigma}^{\mathsf{b}}_{0,k} = \hat{\Sigma}^{\mathsf{b}}_{0,k} \\
\tilde{\Pi}^{\mathsf{b}}_{0,k} &= \hat{\Pi}^{\mathsf{b}}_{0,k} \\
\mathsf{D\bar{d}H}_{i+1,k} &= \mathsf{D\bar{d}H}(B(\breve{\Sigma}^{\mathsf{b}}_{i,k})) \quad \text{(Note: Remark 1 implies close under complement)} \\
\mathsf{D\bar{d}H}^{\tau}_{i+1,k} &= \mathsf{D\bar{d}H}^{\tau}(B(\breve{\Sigma}^{\mathsf{b}}_{i,k})) \quad \text{(Note: Remark 1 implies close under complement)} \\
\tilde{\Sigma}^{\mathsf{b}}_{i,k} &= \mathsf{E}(\mathsf{D\bar{d}H}_{i,k}), \quad \tilde{\Sigma}^{\mathsf{b},\tau}_{i,k} = \mathsf{E}(\mathsf{D\bar{d}H}^{\tau}_{i,k}) \\
\breve{\Sigma}^{\mathsf{b}}_{i,k} &= \mathsf{E}(\widetilde{\mathsf{EuH}}(\tilde{\Sigma}^{\mathsf{b}}_{i,k})), \quad \breve{\Sigma}^{\mathsf{b},\tau}_{i,k} = \mathsf{E}(\widetilde{\mathsf{EuH}}^{\tau}(\tilde{\Sigma}^{\mathsf{b},\tau}_{i,k})) \\
\tilde{\Pi}^{\mathsf{b}}_{i,k} &= \mathsf{U}(\mathsf{D\bar{d}H}_{i,k}), \quad \tilde{\Pi}^{\mathsf{b},\tau}_{i,k} = \mathsf{U}(\mathsf{D\bar{d}H}^{\tau}_{i,k}) \\
\breve{\Pi}^{\mathsf{b}}_{i,k} &= \mathsf{U}(\widetilde{\mathsf{UeH}}(\tilde{\Pi}^{\mathsf{b}}_{i,k})), \text{ and } \breve{\Pi}^{\mathsf{b},\tau}_{i,k} = \mathsf{U}(\widetilde{\mathsf{UeH}}^{\tau}(\tilde{\Pi}^{\mathsf{b},\tau}_{i,k}))
\end{aligned}$$

When reading these complexity classes, it might be helpful to read $\hat{\Sigma}^{\mathsf{b}}_i$ as **prenex** Σ^{b}_i, $\tilde{\Sigma}^{\mathsf{b}}_i$ as **iterable** Σ^{b}_i, and $\breve{\Sigma}^{\mathsf{b}}_i$ as **query iterable** Σ^{b}_i. From the definitions, $\hat{\Sigma}^{\mathsf{b}}_{i,k} \subseteq \tilde{\Sigma}^{\mathsf{b},\tau}_{i,k} \subseteq \breve{\Sigma}^{\mathsf{b},\tau}_{i,k} \subseteq \Sigma^{\mathsf{b}}_{i,k}$

and $\hat{\Pi}^{\mathsf{b}}_{i,k} \subseteq \tilde{\Pi}^{\mathsf{b},\tau}_{i,k} \subseteq \breve{\Pi}^{\mathsf{b},\tau}_{i,k} \subseteq \Pi^{\mathsf{b}}_{i,k}$. When the language L_k is clear, or if it is unimportant, we will drop the subscript k from our notations.

The theory $BASIC_k$ consists of all substitution instances of a finite set of quantifier free axioms for the non-logical symbols of L_k, $k = 1, 2$. These are listed in Buss [2] except for the axioms for MSP, PAD, and $\dot{-}$ which are listed in Takeuti [15]. Proofs in this paper are assumed to be in the sequent calculus system LKB of Buss [2].

Definition 1 *Let τ be a collection of non-decreasing, 0- or 1- ary terms. The Ψ-IND^{τ} axioms are substitution instances of $IND^{\ell,A}$:*

$$A(0) \wedge \forall x < \ell(a)(A(x) \supset A(Sx)) \supset A(\ell(a))$$

where $A \in \Psi$ and $\ell \in \tau$.

Let id denote the identity function. The notations IND and $LIND$ will be used instead of $IND^{\{id\}}$ and $IND^{\{|id|\}}$.

Definition 2 *For $i \geq 0$, we axiomatise:*

1. *T^i_k as $BASIC_k$+$\hat{\Sigma}^{\mathsf{b}}_{i,k}$-$IND$,*
2. *$\breve{T}^i_k$ as $BASIC_k$+$\breve{\Sigma}^{\mathsf{b},\{|id|\}}_{i,k}$-$IND$,*
3. *S^i_k as $BASIC_k$+$\hat{\Sigma}^{\mathsf{b}}_{i,k}$-$LIND$, and*
4. *$\breve{S}^i_k$ as $BASIC_k$+$\breve{\Sigma}^{\mathsf{b},\{|id|\}}_{i,k}$-$LIND$, respectively.*

We define $S_k := \cup_i S^i_k$.

It is known that $S^i_2 = \breve{S}^i_2$ and $T^i_2 = \breve{T}^i_2$ [13]. From Buss [2], $S^i_k \subseteq T^i_k \subseteq S^{i+1}_k$ and this same proof shows $\breve{S}^i_k \subseteq \breve{T}^i_k \subseteq \breve{S}^{i+1}_k$. The theory $I\Delta_0$ is defined using the language 0,S,+,'$\cdot$'; $\leq$. It consists of axioms for these symbols together with Δ_0-IND. The symbols in L_1 are all definable in $I\Delta_0$, and it is known that S_1 is a conservative extension of $I\Delta_0$. Krajíček [8] has more details concerning $I\Delta_0$.

The last definitions needed to present TLS, TLS^i_k, and TSC^i_k are now given.

Definition 3 *For L_k-formulas Ψ where $k > 1$, Ψ-WSN (weak successive nomination rule) is the following rule:*

$$\frac{b \leq |k(j,\vec{a})| \rightarrow \exists! x \leq |k| A(j,\vec{a},b,x)}{\rightarrow \exists w \leq \mathrm{bd}(|k|,t) \forall j < |t| A(j,\vec{a},\hat{\beta}_{|k^*|}(j,w),\hat{\beta}_{|k^*|}(Sj,w))}$$

where $A \in \Psi$ and $\mathrm{bd}(a,b) := 2(2a\#2b)$.

Looking back at the definition of $\mathsf{D\bar{d}H}^{\tau}_{i,k}$, we note that for any (ℓ,ϵ) steppable formula $F_{\vec{t},\vec{B}}$, the formula $\exists C' \leq \ell(t) F_{\vec{t},\vec{B}}(C,C',\vec{a})$ is provable in $BASIC$ as at least one the clauses in $F_{\vec{t},\vec{B}}$ is trivially true, and C' is otherwise computed as a term from C. On the other hand, an (ℓ,ϵ) iteration formula $F_{\vec{t},\vec{B}}$,

$$\begin{aligned}(\exists w \leq 2^{2\cdot m|t^\star_1|})(\forall u \leq \ell^{\epsilon}(t^\star_1))[C \leq \ell(t_1) \wedge \hat{\beta}_{|\ell(t^\star_1)|}(0,w) = C \wedge \\ C' \leq \ell(t_1) \wedge t_2(\hat{\beta}_{|\ell(t^\star_1)|}(\min(\ell^{\epsilon}(t^\star_1),S(c)),w)) = C' \wedge \\ B_1(\hat{\beta}_{|\ell(t^\star_1)|}(\min(u,c),w),\hat{\beta}_{|\ell(t^\star_1)|}(S(\min(u,c)),w),\vec{a})],\end{aligned}$$

asserts the existence of a sequence of values between C and C' each less than or equal to $2^{\ell^{1-\epsilon}(t'_1)}$ that follow from each other according to an (ℓ,ϵ) iterable formula. So although a formula of the form $\exists C' \leq \ell(t) F_{\vec{t},\vec{B}}$ is true in the standard model, it is not necessarily provable in $BASIC_k$. We use such formulas as axioms for the theories we now define. Such formulas are equivalent to formulas in $\mathsf{D\bar{d}H}^{\tau}_{i,k}$ with trivial accept state.

Definition 4 *The* $\mathsf{D\bar{d}H}^{\tau}_{i,k}$*-ITER axioms are the* $\mathsf{D\bar{d}H}^{\tau}_{i,k}$ *formulas with trivial accept state.*

The last axiom needed to define *TLS* is:

Definition 5 *For* L_k*-formulas where* $k > 1$, Ψ*-REPL (quantifier replacement) is the schema:*

$$\frac{\Gamma \to (\forall x \le |s|)(\exists y \le t(x,a))A(x,y,a), \Delta}{\Gamma \to (\exists w \le \mathrm{bd}(t^*(|s|,a),s))(\forall x \le |s|)A(x, \dot{\beta}_{|t^*(|s|,a)|}(x,w),a), \Delta.}$$

where $A \in \Psi$ *and* $\min(x,y) := x + y \dot{-} \max(x,y)$, $\dot{\beta}_{t,s}(x,w) := \min(\hat{\beta}_t(x,w), s)$.

Definition 6 *For* $i, k \ge 1$, *we axiomatize:*

1. *LIOpen*$_k$ *as BASIC*$_k$+open_k*-LIND,*
2. *TLS as LIOpen*$_2$+$\hat{\Sigma}^{\mathsf{b}}_{1,2}$*-WSN*+$\hat{\Sigma}^{\mathsf{b}}_{1,2}$*-REPL,*
3. TLS^i_k *as LIOpen*$_k$+$\mathsf{D\bar{d}H}^{\{p(|id|)\}}_{i,k}$*-ITER, and*
4. TSC^i_k *as LIOpen*$_k$+$\mathsf{D\bar{d}H}^{\{2^{p(||id||)}\}}_{i,k}$*-ITER.*

3 Basic Containments, Closures, and Definability Results

We now develop the basic relationships between TLS^i_k, TSC^i_k, and $\breve{S}^i_k$, and then show how in these theories machine computations can be expressed.

Lemma 2 *For* $i, k \ge 1$, *let* T *be* TLS^i_k *and* τ *be* $\{p(|id|)\}$ *or let* T *be* TSC^i_k *and* τ *be* $2^{p(||id||)}$. *Then* T *proves for each* $A \in \mathsf{D\bar{d}H}^{\tau}_{i-1,k}$ *that there is a formula* $A^0 \in \mathsf{D\bar{d}H}^{\tau}_{i-1,k}$ *such that* T *proves* $\neg A \Leftrightarrow A^0$.

Proof. Let $A(i,\vec{a})$ be in $\mathsf{D\bar{d}H}^{\tau}_{i-1,k}$. If A's accept state is given by the clause $t'(C') = 1$, let A^0 be the formula obtained from A replacing this clause with $1 \dot{-} t'(C') = 1$. So $\neg A^0$ is logically equivalent to A. □

Definition 7 Ψ*-COMP are the bit comprehension axioms* $COMP_A := \exists w \le t \forall i < |t|(\mathrm{BIT}(i,w) = 1 \Leftrightarrow A(i,\vec{a}))$ *for formulas* $A \in \Psi$.

Lemma 3 *For* $i, k \ge 1$, *let* T *be* TLS^i_k *and* τ *be* $\{p(|id|)\}$ *or let* T *be* TSC^i_k *and* τ *be* $2^{p(||id||)}$. *Then* T *proves: (a)* $\mathsf{D\bar{d}H}^{\tau}_{i,k}$*-COMP, (b)* $\mathsf{D\bar{d}H}^{\tau}_{i,k}$*-LIND.*

Proof. For (a), let $A(i,\vec{a})$ be in $\mathsf{D\bar{d}H}^{\tau}_{i,k}$. Consider the formula $C(i,\vec{a},b,x)$:

$$(\min(0,1) = x \wedge A^0(i,\vec{a}) \wedge b = b) \vee ((\min(1,1) = x \wedge A(i,\vec{a}) \wedge b = b) \wedge \\ (\min(0,1) \neq x \vee \neg A^0(i,\vec{a}) \vee b \neq b)).$$

This formula is a (1, 1) steppable $\mathsf{D\bar{d}H}^{\tau}_{i,k}$-formula. To see this, the formula itself has the shape of a steppable formula, and if $\ell = 1, \epsilon = 1$, then $|1| \le \ell^{1-1}(h(x)) = 1$ for any term h. Therefore, T can use a $\mathsf{D\bar{d}H}^{\tau}_{i,k}$-*ITER* axiom to prove

$$\exists w \le 2^{|t|+1} \forall j \le |t| C(j,\vec{a},\mathrm{BIT}(j,w),\mathrm{BIT}(Sj,w))$$

From the definition of C in terms of A, *BASIC* proves $C(i,\vec{a},b,1) \Leftrightarrow A(i,\vec{a})$. From this, it follows in T that for a w satisfying the above, that $v = \lfloor \frac{1}{2} w \rfloor$ satisfies $\forall i < |t|(\mathrm{BIT}(i,v) = 1 \Leftrightarrow A(i,\vec{a}))$, so T proves $COMP_A$.

For (b), from (1) if A is $\mathsf{D\bar{d}H}^{\tau}_{i,k}$, then TLS^i_k proves *LIND* for A since T proves $COMP_A$ and since T proves *LIND* for the formula $\mathrm{BIT}(i,y) = 1$. □

A formula A is said to be $\hat{\Delta}^b_i$ *(resp.* Δ^b_i, $\tilde{\Delta}^{b,\tau}_i$*) in a theory* T if $T \vdash A^\Sigma \equiv A \equiv A^\Pi$ where A^Σ is $\hat{\Sigma}^{\mathsf{b}}_i$ (resp. Σ^{b}_i, $\tilde{\Sigma}^{\mathsf{b},\tau}_i$) and A^Π is $\hat{\Pi}^{\mathsf{b}}_i$ (resp. Π^{b}_i, $\tilde{\Pi}^{\mathsf{b},\tau}_i$). A multifunction f is Ψ*-defined in* T if there is an $A \in \Psi$ such that $T \vdash \forall x \exists y A(x,y)$ and $\mathbb{N} \models A(x, f(x))$. For f to be Ψ-defined as a function, we further require $T \vdash \forall x \exists! y A(x,y)$. For an L_2 theory with quantifier replacement for $\hat{\Sigma}^{\mathsf{b}}_i$-formulas, the notions of (Σ^{b}_i, $\breve{\Sigma}^{\mathsf{b}}_i$, $\tilde{\Sigma}^{\mathsf{b}}_i$, $\tilde{\Sigma}^{\mathsf{b},\tau}_i$, $\hat{\Sigma}^{\mathsf{b}}_1$)-definability coincide; similarly, the notions Δ^b_1 and $\hat{\Delta}^b_1$-coincide (notice $\hat{\Sigma}^{\mathsf{b}}_i \subset \mathsf{ED\bar{d}H}^{\tau}_i$ and $\hat{\Pi}^{\mathsf{b}}_i \subset \mathsf{UD\bar{d}H}^{\tau}_i$).

To give a flavor of the arguments that can be carried out in TLS^1_1, we prove the following definability results.

Proposition 1 *TLS^1_1 can $\mathsf{D\bar{d}H}^{\{p(|id|)\}}_1$ define a function that counts the number of on bits in v.*

For this purpose, we use step and iteration functions that operate on values encoding

$$\langle\langle \text{block\_num}, \text{offset}, \text{cnt} \rangle\rangle.$$

We imagine the bit positions of v are split into blocks of $\ell^{1/2}(v)$ bits. Here block_num represents which block of $\ell^{1/2}(v)$ bits of the v we are currently counting, offset represents a position in that block, and cnt represents the number of 1 bits in the bit positions less than or equal to block_num $\cdot\, \ell^{1/2}(v) +$ offset in all of v. Let $t_1(C) :=$

$$\langle\langle \{C\}_1, \{C\}_2, \{C\}_3 + \mathrm{BIT}(\{C\}_2 + \{C\}_1 \cdot \ell^{1/2}(v), v) \rangle\rangle$$

and let B_1 be the formula $C = C$. Let $B'_1 := F_{t_1,B_1}$ where F_{t_1,B_1} is the steppable formula corresponding to t_1 and B_1. Assuming $\{C\}_3$ holds the correct count of '1' bit positions less than $D := \{C\}_2 + \{C\}_1 \cdot \ell^{1/2}(v)$ then $\{t_1(C)\}_3$ will contain the count for bit positions less than or equal to D. Let $t'_1 := 2^{8\cdot(||v||+1)(\ell^{1/2}(v)+1)}$. Here t'_1 can be used to bound the size of any sequence of $\ell^{1/2}(v)$ many triples of 3 values less than or equal to $||v||$. Let $t'_2(C) := \langle\langle \{C\}_1 + \ell^{1/2}(t'_1), \{C\}_2, \{C\}_3 \rangle\rangle$ and let $F_{\vec{t'},B'}(C, C', c, v) := \mathrm{Iter}_{t'_1,t'_2,B'_1}(C, C', c, v)$. So $TLS^1_1 \vdash \exists C' \leq |v| F_{\vec{t'},B'}(C, C', c, v)$ as it is a $\mathsf{D\bar{d}H}^{\{p(|id|)\}}_i$-*ITER* axiom. Using $\hat{\Pi}^{\mathsf{b}}_0$ *LIND* on c (available by Lemma 3), one can show that if w and w' both satisfy the formula within the existential of $F_{\vec{t'},B'}(C, C', |v|, v)$ then $w = w'$, this in turn shows $TLS^1_1 \vdash \exists! C' \leq |v| F_{\vec{t'},B'}(C, C', |v|, v)$. We can now repeat this process. Set $B''_1(C, C', v) := F_{\vec{t'},B'}(C, C', |v|, v)$, $t''_1(C) = t'_1(C)$, and $t''_2(C) = \{C\}_3$. Define

$$Numones(v, C') := \mathrm{Iter}_{t''_1,t''_2,B''_1}(\langle\langle 0,0,0 \rangle\rangle, C', |v|, v).$$

Then using a $\mathsf{D\bar{d}H}^{\{p(|id|)\}}_1$-*ITER* axiom,

$$TLS^1_1 \vdash \exists C' \leq |v| Numones(v, C').$$

Using $\hat{\Pi}^{\mathsf{b}}_0$ *LIND* together with the uniqueness just shown for the $\ell^{1/2}$-length sub-computations, TLS^1_1 can prove C' unique and its value computes the number on 1 bits in v.

Proposition 2 *TLS proves the TLS^1_2 axioms.*

Proof. (Sketch) Using $\hat{\Sigma}^{\mathsf{b}}_{1,2}$-*REPL*, any $\mathsf{D\bar{d}H}^{\{p(|id|)\}}_{1,2}$-formula is provably equivalent to a $\hat{\Sigma}^{\mathsf{b}}_{1,2}$ formula. Inductively, using $\hat{\Sigma}^{\mathsf{b}}_{1,2}$-*WSN* starting from steppable formulas, TLS then proves any individual $\mathsf{D\bar{d}H}^{\{p(|id|)\}}_{1,2}$-*ITER* axiom. □

Proposition 3 *For $i, k > 0$, $TLS^i_k \subseteq TSC^i_k \subseteq \breve{S}^i_k \subseteq TLS^{i+1}_k$.*

Proof. $TLS^i_k \subseteq TSC^i_k$ follows directly from the definitions. Next note that $\mathsf{D\bar{d}H}^{\{2^{p(||id||)}\}}_i$ consists of those formulas in $\cup_m (\mathsf{D\bar{d}})^{\{2^{p(||id||)}\}}_m (B(\breve{\Sigma}^{\mathsf{b}}_i))$ with accept state. Let **iteration complexity** m of an $\mathsf{D\bar{d}H}^{\{2^{p(||id||)}\}}_i$-*ITER* axiom be the least m such that its $\mathsf{D\bar{d}H}^{\{2^{p(||id||)}\}}_i$ formula is in $(\mathsf{D}\bar{\mathsf{d}})_m^{\{2^{p(||id||)}\}}(B(\breve{\Sigma}_i^{\mathsf{b}}))$. We prove $TSC_k^i \subseteq \breve{S}_k^i$ by induction on the iteration complexity of the $\mathsf{D}\bar{\mathsf{d}}\mathsf{H}_i^{\{2^{p(||id||)}\}}$ formula used in a $\mathsf{D}\bar{\mathsf{d}}\mathsf{H}_i^{\{2^{p(||id||)}\}}$-*ITER* axiom. When $m = 1$, the $\mathsf{D}\bar{\mathsf{d}}\mathsf{H}_i^{\{2^{p(||id||)}\}}$-*ITER* axiom is based on a steppable formula, $F_{\vec{t},\vec{B}}(C, C', \vec{a})$. As we require one of the B_i formulas in such a disjunction to be trivially true, *BASIC* can prove $\exists C' \leq \ell F_{\vec{t},\vec{B}}(C, C', \vec{a})$ for ℓ the growth rate used in the axiom. Suppose we have an $\mathsf{D}\bar{\mathsf{d}}\mathsf{H}_i^{\{2^{p(||id||)}\}}$-*ITER* axiom

$$F'(c) := \exists C' \leq \ell(r)\mathrm{Iter}_{t_1,t_2,B_1}(C, C', c, \vec{a})$$

with iteration complexity $m + 1$ where $\ell \in \{2^{p(||id||)}\}$ and t_1 is an L_k term. So B_1 is in $(\mathsf{D}\bar{\mathsf{d}})_m^{\{2^{p(||id||)}\}}(B(\breve{\Sigma}_i^{\mathsf{b}}))$. By the induction hypothesis, $\breve{S}_k^i$ proves $\exists D' \leq \ell(r)B_1(D, D', \vec{a})$. Notice $F'(0)$ is equivalent to B_1 and that $F(c) \supset F(S(c))$ follows by concatenating on to w, a D' witnessing $B_1(\hat{\beta}_{|\ell(t^\star)|}(c, w), D', \vec{a})$. Therefore, as $\mathrm{Iter}_{t_1,t_2,B_1}(C, C', c, \vec{a}) \in \mathsf{D}\bar{\mathsf{d}}\mathsf{H}_i^{\{2^{p(||id||)}\}} \subseteq \tilde{\Sigma}_i^{\mathsf{b},\{2^{p(||id||)}\}}$, $\breve{S}_k^i$ proves $F'(\ell^\epsilon(s))$. This entails the axiom for all c. So $TSC_k^i \subseteq \breve{S}_k^i$. Finally, the last inclusion follows from Lemma 3 as $\breve{\Sigma}_i^{\mathsf{b},\{|id|\}} \subseteq \mathsf{D}\bar{\mathsf{d}}\mathsf{H}_{i+1,k}^{\{|id|\}}$. □

We now fix our formalization of an m-work tape oracle Turing Machine M. To access its input, we assume M writes i on the first work tape, enters a state q_{in}, and in one time step the ith symbol of the input appears under the tape head. M also has distinguished states q_{start}, q_{accept}, q_{reject}, q_{query}, q_{yes}, q_{no}, corresponding to the start state of a computation, the accepting, rejecting halt states at the end of computation, a query to the oracle state, a 'yes' response to the query state, and a 'no' response to the query state. We write $\ulcorner q \urcorner$ (Gödel code of q) to refer to the state q encoded as a natural number. Oracle queries consist of the input x and the oracle tape contents. A configuration of M is a tuple:

$$\langle\langle \mathrm{q, ipos, qcnt, ycnt, lotape, rotape, ltape_1, rtape_1, \ldots, ltape}_m, \mathrm{rtape}_m \rangle\rangle$$

Here q is the current state of machine, *ipos* is the input tape position; *qcnt* is the total count of queries made so far; *ycnt* is the number of times M went into the q_{yes} state; *lotape* are the oracle tape contents to the left of and including the tape head; *rotape* are the oracle tape contents to the right of the tape head; similarly, $ltape_i$ are the contents of the ith work tape contents to the left of and including the tape head; $rtape_i$ are the ith work tape contents to the right of the tape head. We write *Start* and *Reject* for the terms $\langle \ulcorner q_{start} \urcorner, 0, \ldots, 0\rangle$ and $\langle \ulcorner q_{reject} \urcorner, 0, \ldots, 0\rangle$, representing start and a reject final configurations. *Reject* is the only valid configuration that can follow a C which does not code for a configuration. Let $IsConfig_M(C)$ be the open formula which uses *ispair* and projections to check that C is a $m + 6$ tuple and that $\{C\}_1$ is a state of machine M. If M has a space bound s, then it also checks that the Gödel code of the configuration is at most what would be needed for this space bound and that $QCount(C) \leq s \wedge YesCount(C) \leq s$. We write $IsQuery(C), IsYes(C)$ for $\{C\}_1 = \ulcorner q_{query} \urcorner$ and $\{C\}_1 = \ulcorner q_{yes} \urcorner$. We write $Query(C)$ for $\{C\}_5$. We write $QCount(C)$ and $YesCount(C)$ for $\{C\}_3$ and $\{C\}_4$ that return *qcnt* and *ycnt*.

The L_1 term $Next_M(C, x, y)$ uses *cond* to compute C' as follows:

1. If $\neg IsConfig_M(C)$ then return *Reject*.
2. if $\{C\}_1 = \ulcorner q_{accept} \urcorner$ or $\{C\}_1 = q_{reject}$ then return C.
3. If $\neg\{C\}_1 = \ulcorner q_{query} \urcorner$ then return a C' that follows in one step from C according to M given input x and $QCount(C') = QCount(C)$.
4. If $\{C\}_1 = \ulcorner q_{query} \urcorner$ then return a C' is that either in the state y if $y = \ulcorner q_{yes} \urcorner$ or $\ulcorner q_{no} \urcorner$ otherwise. Further C' has $QCount(C') = QCount(C') + 1$, and C' is otherwise obtained from C according to one step of M on x . If C' is in the state $y = \ulcorner q_{yes} \urcorner$, then have $YesCount(C') = YesCount(C) + 1$ and otherwise, $YesCount(C') = YesCount(C)$. $Next_M$ does not check correctness of the oracle response.

Let $0 < \epsilon \leq 1$. Assume ℓ is a unary term satisfying $|\ell(z)| \leq m' \cdot \ell^{1-\epsilon}(h(z))$ for some L_k term h and $m' \in \mathbb{N}$. We inductively define two formulas $Comp^n_{M,A,\epsilon,\ell}$ and $MComp^n_{M,A,\epsilon,\ell}$ designed to check the correctness of computation sequences of $\ell(x)$ bounded configurations of M on x of length $(\ell^\epsilon(x))^n$, but which vary in how tightly the correctness checks for oracle A responses are done:

$$Comp^0_{M,A,\epsilon,\ell}(C,C',x) := F_{\vec{t},\vec{B}}(C,C',x), \text{ steppable via}$$
$$t_1 := Next_M(C,x,\ulcorner q_{yes}\urcorner),\ t_2 := Next_M(C,x,\ulcorner q_{no}\urcorner) \text{ and } B_1 := B_2 :=$$
$$(IsQuery(C) \supset (IsYes(C') \Leftrightarrow A(Query(C),x))$$
$$Comp^{n+1}_{M,A,\epsilon,\ell}(C,C',x) := \exists v \le 2^{|h|}\forall i < \ell^\epsilon[\hat{\beta}_{|\ell|}(0,v) = C \wedge$$
$$\hat{\beta}_{|\ell|}(\ell^\epsilon(x),v) = C' \wedge Comp^n_{M,A,\epsilon,\ell}(\hat{\beta}_{|\ell|}(i,v),\hat{\beta}_{|\ell|}(Si,v),x)]$$

$$MComp^0_{M,A,\epsilon,\ell}(C,C',x,qres) := F_{\vec{t},\vec{B}}(C,C',x) \wedge (IsYes(C') \supset A(Query(C),x)$$
where $F_{\vec{t},\vec{B}}(C,C',x,qres)$ is steppable using
$$t_1 := Next_M(C,x,\ulcorner q_{yes}\urcorner),\ t_2 := Next_M(C,x,\ulcorner q_{no}\urcorner)$$
$$\text{and } B_1 := B_2 := (IsQuery(C) \supset (IsYes(C') \Leftrightarrow BIT(QCount(C),qres) = 1)$$
$$MComp^{n+1}_{M,A,\epsilon,\ell}(C,C',x,qres) := \exists v \le 2^{|h|}\forall i < \ell^\epsilon[\hat{\beta}_{|\ell|}(0,v) = C \wedge$$
$$\hat{\beta}_{|\ell|}(\ell^\epsilon(x),v) = C' \wedge MComp^n_{M,A,\epsilon,\ell}(\hat{\beta}_{|\ell|}(i,v),\hat{\beta}_{|\ell|}(Si,v),x,qres)]$$

If $A \in \Psi$, $Comp^0_{M,A,\epsilon,\ell}(C,C',x)$ is (ℓ,ϵ) is steppable in $\mathsf{D\bar{d}H}((B(\Psi))$ and $MComp^0_{M,A,\epsilon,\ell}(C,C',x)$ is (ℓ,ϵ) steppable in $\widetilde{\mathsf{EuH}}(\Psi)$. We notice if not $IsQuery(C)$ then B_1 trivially holds for $Comp^0_{M,A,\epsilon,\ell}$ and for $MComp^0_{M,A,\epsilon,\ell}$ and so a C' satisfying either of these formulas would need to be computed according to (3) from our definition of $Next_M(C,x,y)$. In the case where $IsQuery(C)$ holds for the $Comp^0_{M,A,\epsilon,\ell}$, B_1 and B_2 ensure that the state in C' is $\ulcorner q_{yes}\urcorner$ or $\ulcorner q_{no}\urcorner$ based on whether $A(Query(C),x)$ holds. $Next_M$ would compute C' according to condition (4). $MComp^0_{M,A,\epsilon,\ell}$ modifies this so that rather than look at $A(Query(C),x)$, instead the state in C' is $\ulcorner q_{yes}\urcorner$ or $\ulcorner q_{no}\urcorner$ based on whether the $QCount(C)$th query bit of $qres$ is 1. If the state of C' is $\ulcorner q_{yes}\urcorner$, then for $MComp^0_{M,A,\epsilon,\ell}$ to hold, the clause $(IsYes(C') \supset A(Query(C),x)$ implies $A(Query(C),x)$ must hold, so 'yes' responses must be correct. The inductive definition of $Comp^{n+1}$ (resp. $MComp^{n+1}$) from $Comp^n$ (resp. $MComp^n$) can be modified into an iteration formula by adding clauses bounding the size of C and C', so $Comp^n_{M,A,\epsilon,\ell}$ is also in $\mathsf{D\bar{d}H}((B(\Psi))$ and $MComp^n_{M,A,\epsilon,\ell}$ is also in $\widetilde{\mathsf{EuH}}(\Psi)$. So if $A \in \tilde{\Sigma}^{\mathsf{b}}_i$, $Comp^n_{M,A,\epsilon,\ell}$ is equivalent to a $\mathsf{D\bar{d}H}^{\{\ell\}}_{i+1}$ formula and $MComp^n_{M,A,\epsilon,\ell}$ is equivalent to a $\breve{\Sigma}^{\mathsf{b},\{\ell\}}_i$ formula. These formulas hold if there are ℓ^ϵ length sequences of $Comp^{n-1}_{M,A,\epsilon,\ell}$ (resp. $MComp^{n-1}_{M,A,\epsilon,\ell}$) satisfying steps that take configuration C to configuration C'.

Theorem 1 *For $k = 1,2$ and $i \ge 1$: (a) Each $L \in \mathsf{L}$ is represented by a $\tilde{\Delta}^{b,\{|id|\}}_1$ predicate in TLS^1_k. (b) Each $L \in \mathsf{SC}$ is represented by a $\tilde{\Delta}^{b,\{2^{p(||id||)}\}}_1$ predicate in TSC^1_k . (c) Each $L \in \mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}$ is represented by a $\tilde{\Delta}^{b,\{|id|\}}_{i+1}$ predicate in S^i_k. (d) Each $L \in \mathsf{SC}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}$ is represented by a $\tilde{\Delta}^{b,\{2^{p(||id||)}\}}_{i+1}$ predicate in TSC^{i+1}_k.*

Proof. We prove these in the order (a), (d), (c) which is roughly in the order of difficulty. We skip the proof of (b) as (a) and (b) are proven in the same way except for size of the bounds in the $ITER$ axioms used. For (a), fix $L \in \mathsf{L}$. So $L \in \mathsf{TISP}[n^{m'}, m\log n]$ for some fixed m' and m, and $n = |x|$. Let $M(x)$ recognize L. So any configuration C of M has $C \le |x|^{m''}$ for some constant m''. Let $D(v,x) := v \ge v \vee x \ge x$. This dummy formula is provable in $BASIC$. We note $LIOpen$ proves

$$\forall C\exists! C' \le |x|^{m''} Comp^0_{M,D,1/2,|x|^{m''}}(C,C',x)$$

as a next configuration of M, C', after C involves fixed tape and C manipulations. As $\exists C' \le |x|^{m''} Comp^n_{M,A,1/2,|x|^{m''}}(C,C',x)$ is a $\mathsf{D\bar{d}H}^{\{p(|id|)\}}_1$ $ITER$ axiom, TLS^1_k proves the existence of a

C' for each $n > 0$ that is the result of a computational sequence of length $\ell^{1/2}(x) = 2^{1/2||x||} \approx |x|^{1/2}$. Arguing by induction on n, we have already argued uniqueness in the $n = 0$ case using *LIOpen* that both C' and the outer existential asserting configurations after every $\ell^{1/2}(x) - 1$ steps is unique. Using *LIOpen* on the w asserted by the $\mathsf{D\bar{d}H}_1^{\{p(|id|)\}}$*ITER* axiom also proves C' and the outermost existential sequence are unique for the $n+1$ case, showing the induction holds. So using $2m'+1$ applications of $\mathsf{D\bar{d}H}_{1,k}^{\{p(|id|)\}}$-*ITER* and an induction argument to show the uniqueness, TLS_2^1 proves $\forall C \exists! C' \leq |x|^{m''} Comp^{2m'+1}_{M,D,1/2,|x|^{m''}}(C, C', x)$, and C' will correspond to a final state of an execution of M on x. So L can be represented as either $A^\Sigma \in \tilde{\Sigma}_1^{\mathsf{b},\{p(|id|)\}}$ defined as

$$\exists C' \leq |x|^{m''}[Comp^{2m'+1}_{M,D,1/2,|x|^{m''}}(Start, C', x) \wedge ((C'))_1 = \ulcorner q_{accept} \urcorner].$$

or $A^\Pi \in \tilde{\Pi}_1^{\mathsf{b},\{p(|id|)\}}$ defined as

$$\neg\exists C' \leq |x|^{m''}[Comp^{2m'+1}_{M,D,1/2,|x|^{m''}}(Start, C', x) \wedge \neg((C'))_1 = \ulcorner q_{accept} \urcorner].$$

Since TLS_k^1 proves a C' satisfying $Comp^{2m'+1}_{M,D,1/2,|x|^{m''}}(Start, C', x)$ is unique, $TLS_k^1 \vdash A^\Sigma \Leftrightarrow A^\Pi$.

The proof of (d) is similar. Suppose $L \in \mathsf{SC}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}$, let M with oracle $A \in \tilde{\Sigma}^{\mathsf{b}}_{i,k}$ recognize L. Since M is an $\mathsf{SC} = \mathsf{TISP}[\mathsf{poly}, \mathsf{polylog}]$ oracle machine, each configuration C of M has a value bounded by $2^{||x||^{m''}}$ for some fixed m''. As before, *LIOpen* proves $\forall C \exists! C' \leq 2^{||x||^{m''}} Comp^0_{M,A,1/2,2^{||x||^{m''}}}(C, C', x)$. This time $Comp^n_{M,A,1/2,2^{||x||^{m''}}}$ is a $\tilde{\Sigma}^{\mathsf{b},\{2^{p(||id||)}\}}_{i+1}$ formula, and TSC_k^{i+1} can use $2m'+1$ $\mathsf{D\bar{d}H}^{\{2^{p(||id||)}\}}_{i+1,k}$-*ITER* axioms and *open*-*LIND* to prove

$$\forall C \exists! C' \leq 2^{||x||^{m''}} Comp^{2m'+1}_{M,A,1/2,2^{||x||^{m''}}}(C, C', x).$$

The rest of the argument proceeds as before to get $\tilde{\Sigma}^{\mathsf{b},\{2^{p(||id||)}\}}_{i+1}$-formula A^Σ and $\tilde{\Pi}^{\mathsf{b},\{2^{p(||id||)}\}}_{i+1}$-formula A^Π for L that $TSC_k^1 \vdash A^\Sigma \Leftrightarrow A^\Pi$.

To prove (c), let $L \in \mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}$. Let M with oracle $A \in \tilde{\Sigma}^{\mathsf{b}}_{i,k}$ recognize L. Assume M runs in time bounded by $|x|^{m'}$ and, as it uses logspace, that codes for configurations are less than $|x|^{m''}$. This also entails it makes fewer than $m'' \cdot ||x||$ queries and our definition of $IsConfig(C)$ would return false if C had a $QCount$ or $YesCount$ higher than this. Let $MC'(C, C', b) :=$

$$\exists qres \leq 2^{m'' \cdot ||x||} MComp^{2m'+1}_{M,A,1/2,|x|^{m''}}(C, C', x, qres) \wedge YesCount(C') = b)$$

and define $MC(C, b) := \exists C' \leq |x|^{m''} MC'(C, C', b)$. If $b = 0$, $MComp^{2m'+1}_{M,A,1/2,|x|^{m''}}$ is equivalent to a $\breve{\Sigma}_1^{\mathsf{b},\{p(|id|)\}}$-formula. It makes no assertion about oracle responses being correct and so $\breve{S}_k^1$ can prove using $\breve{\Sigma}_1^{\mathsf{b},\{p(|id|)\}}$-*LIND* that $MC(0)$. $MC(b)$, in general, is a $\breve{\Sigma}_i^{\mathsf{b},\{|id|\}}$-formula. Since M makes fewer than $m'' \cdot ||x||$ queries, we have $\neg MC(C, m'' \cdot ||x||)$. Thus, by $\breve{\Sigma}_1^{\mathsf{b},\{p(|id|)\}}$-*LIND*, $\breve{S}_k^i$ proves there exists a b such that $MC(C, b) \wedge \neg MC(C, Sb)$. Since the 'yes' answered queries must be correct, for this b, the 'no' answered queries must also be correct or a 'no' could be switched to a 'yes' implying $MC(Sb)$. C' can be argued to be unique as before, using the uniqueness of C' and outermost existential produced by $MComp^n$ to argue uniqueness for $MComp^{n+1}$. Let A^Σ be the formula

$$\exists C' \leq |x|^{m''} MC'(Start, C', b) \wedge \{C'\}_1 = \ulcorner q_{accept} \urcorner \wedge \neg MC(Start, Sb).$$

This is the existential quantifier followed of a boolean combination of $\breve{\Sigma}_i^{\mathsf{b},\{p(|id|)\}}$-formulas, so will be $\tilde{\Sigma}^{\mathsf{b},\{p(|id|)\}}_{i+1}$. Similarly, if we let A^Π be the formula

$$\neg\exists C' \leq |x|^{m''} MC'(Start, C', b) \wedge \neg\{C'\}_1 = \ulcorner q_{accept} \urcorner \wedge \neg MC(Start, Sb)$$

it is after pushing negations inward a $\tilde{\Pi}^{\mathsf{b},\{p(|id|)\}}_{i+1}$ formula. From uniqueness of C', we can argue $S^i_k \vdash A^\Sigma \Leftrightarrow A^\Pi$. $\square$

Definition 8 *The W-operator on A, $(Wy \leq t)A(x,y)$, returns $y \leq t$ such that $A(x,y)$ holds, if such a y exists, and $t+1$ otherwise. The μ-operator on A, $(\mu y \leq t)A(x,y)$, returns the least y such that $A(x,y)$ holds, if such a y exists, and $t+1$ otherwise.*

Definition 9 *Given languages classes $\mathcal{C}$ and $\mathcal{O}$, define L_k-$F\mathcal{C}[O, wit]$ to be multifunctions f for which there are $L_f \in \mathcal{C}$, $A^f_1, \ldots A^f_m \in \mathcal{O}$, and L_k-terms t_f and $s_1, \ldots, s_m$, such that for each y with $f(\vec{x}) = y$, there is at least one $\vec{z}$ with*

$$z_1 = (Wv_1 \leq s_1(\vec{x}))A^f_1(v_1, \vec{x}), \ldots, z_m = (Wv_m \leq s_m(\vec{x}))A^f_m(v_m, \vec{x})$$

satisfying for all $i \leq |t_f|$, $\mathrm{BIT}(i,y) = 1 \Leftrightarrow \langle i, \vec{x}, \vec{z}\rangle \in L_f$. We write L_k-$F\mathcal{C}$ for the functions in L_k-$F\mathcal{C}[O, wit]$ that arise if the witness oracle list is empty.

We abbreviate L_k-$F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}[\tilde{\Sigma}^{\mathsf{b}}_{i,k}, wit]$ as L_k-$F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}[wit]$ and L_k-$F\mathsf{SC}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}[\tilde{\Sigma}^{\mathsf{b}}_{i,k}, wit]$ as L_k-$F\mathsf{SC}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}[wit]$.

Corollary 1 *For $k = 1, 2$ and $i \geq 1$: (a) TLS^1_k can $\tilde{\Sigma}^{\mathsf{b},\{p(|id|)\}}_1$-define any L_k-$F\mathsf{L}$ function. (b) TSC^1_k can $\tilde{\Sigma}^{\mathsf{b},\{2^{p(||id||)}\}}_1$-define any L_k-$F\mathsf{SC}$ function. (c) S^i_k can $\tilde{\Sigma}^{\mathsf{b},\{p(|id|)\}}_{i+1}$-define any L_k-$F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}[wit]$ multifunction. (d) TSC^{i+1}_k can $\tilde{\Sigma}^{\mathsf{b},\{2^{p(||id||)}\}}_i$-define any L_k-$F\mathsf{SC}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}[wit]$ multifunction.*

Proof. (a) and (b) do not involve witnesses. Suppose f is a L_k-$F\mathsf{L}$ (resp. L_k-$F\mathsf{SC}$) function given by the language $\langle j, \vec{x}\rangle \in L_f$. Let machine M recognizing L_f. Then $A(i, \vec{x}) :=$

$$\exists C' \leq |h(j,\vec{x})|^{m''}[Comp^{2m'+1}_{M,D,1/2,|h|^{m''}}(Start, C', j, \vec{x}) \wedge ((C'))_1 = \ulcorner q_{accept} \urcorner],$$

from the proof of Theorem 1 holds iff $\langle j, \vec{x}\rangle \in L_f$. This formula is in $\mathsf{D\bar{d}H}^{\tau}_{1,k}$where τ is $\{|id|\}$ (resp. $\{2^{p(||id||)}\}$). So by Lemma 3, TLS^1_k (resp. TSC^1_k) proves

$$\exists v \leq 2^{|t|} \forall i < |t|(\mathrm{BIT}(i,v) = 1 \Leftrightarrow A(i, \vec{a})).$$

For both (c) and (d), let τ be either $\{|id|\}$ (for (c)) or $\{2^{p(||id||)}\}$ (for (d)) . Given $C \in \tilde{\Sigma}^{\mathsf{b}}_{i,k}$ define the formula $WQuery_C(x,y,z)$ to be:

$$(C(x,y) \wedge y \leq z) \vee \neg(\exists y' \leq z)(C(x,y') \wedge y = z+1).$$

Using excluded middle, $BASIC$ proves $\forall x \exists y \leq z + 1 WQuery_C(x,y,z)$ and this shows $(Wy \leq z)C(x,y)$ is $B(\tilde{\Sigma}^{\mathsf{b}}_i)$-definable in $BASIC$ and hence in S^i_k or TSC^{i+1}_k. Fix T to be S^i_k or TSC^{i+1}_k and $\mathcal{F}$ to be the corresponding L_k-$F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}[wit]$ or L_k-$F\mathsf{SC}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}[wit]$. Let $f(\vec{x}) \in \mathcal{F}$ be defined via language L_f and oracle languages $A^f_1, \ldots, A^f_m$. As with the proof of (a) and (b) above, the proof of Theorem 1 (d), shows in the TSC^{i+1}_k case that L_f is computed by a $\mathsf{D\bar{d}H}^{\tau}_{1,k}$-predicate $B_f(j, \vec{x}, \vec{z})$. So by Lemma 3 and its ability to compute $WQuery_C$ for $C \in \tilde{\Sigma}^{\mathsf{b}}_{i,k}$, TSC^{i+1}_k proves

$$\begin{aligned} &\forall \vec{x} \exists z_1 \leq s_1(\vec{x}) + 1 WQuery_{A^f_1}(\vec{x}, s_1(\vec{x}), z_1) \wedge \ldots \wedge \\ &\quad \exists z_m \leq s_m(\vec{x}) + 1 WQuery_{A^f_m}(\vec{x}, s_m(\vec{x}), z_m) \wedge \\ &\quad \exists y \leq 2^{|t_f|} \forall j < |t_f|(\mathrm{BIT}(j,y) = 1 \Leftrightarrow B_f(j, \vec{x}, \vec{z})). \end{aligned} \tag{1}$$

Reordering the existentials so that $\exists y \leq 2^{|t_f|}$ is the outermost shows case (d) of the Corollary. For case (c), let M_f compute L_f. Modify M_f to make M'_f which: Takes the coordinate j of the input to L_f for and copies it to a new tape. It then cycles through all bit positions $j' \leq |t_f|$ and computes M_f for that value j', reusing space, and asking the oracle queries needed as it goes, when it gets to value j after performing the simulation it remembers on the new tape whether the machine accepted or rejected but continues simulating M_f for the remaining j'. Finally, this machine accepts or rejects based on the stored accepting or rejection for the jth bit saved on the auxiliary tape. This machine computes the same language as M_f, however, the queries it asks are the same regardless of which bit position is j is being asked for. Consider the formula $FC(\vec{x}, \vec{z}, b)$:

$$\exists w \leq 2^{|t_f|} \forall i \leq |t_f| [\mathrm{BIT}(i, w) = 1 \supset \exists C' \leq |x|^{m''} MC'(Start, C', \mathrm{MSP}(b, ||w||)) \wedge ((C'))_1 = \ulcorner q_{accept} \urcorner \wedge Numones(w, \mathrm{LSP}(b, ||w||))].$$

As $Numones$ is a $\tilde{\Sigma}_1^{\mathsf{b}}$-formula and MC' is $\tilde{\Sigma}_i^{\mathsf{b},\{2^{p(||id||)}\}}$, $FC(b)$ is $\tilde{\Sigma}_i^{\mathsf{b},\{2^{p(||id||)}\}}$ formula. Let FC' be the formula inside the scope of the outermost existential. $\mathrm{MSP}(b, ||w||)$ represents the number of times M_f used a 'yes' answered query in its computation and $\mathrm{LSP}(b, ||w||)$ represents a lower bound on the bit positions $i \leq |t_f|$ on $\vec{x}$ using $\vec{z}$ that were in L_f as w bit value must be correct for 1 positions. Since for all $i \leq |t_f|$ the queries made to the oracle are the same, maximizing $\mathrm{MSP}(b, ||w||)$ together with maximizing $\mathrm{LSP}(b, ||w||)$, hence maximizing b, will correspond to a correct computation using the oracle for all bit positions $|t_f|$. Using $\tilde{\Sigma}_i^{\mathsf{b},\{2^{p(||id||)}\}}$-$LIND$, $\breve{S}_k^i$ proves $\exists w \leq 2^{|t_f|} FC'(\vec{x}, \vec{z}, b) \wedge \neg FC(\vec{x}, \vec{z}, b)$. Replacing the last conjunction in formula (1) with this formula and reordering the existential so that w is the outermost completes the proof of part (c). □

We conclude this section by showing TSC_k^i, TLS_k^i, $\breve{S}_k^{i-1}$ prove various closure properties of the L_k-$F\mathsf{SC}^{\tilde{\Sigma}_{i-1,k}^{\mathsf{b}}}[wit]$ and L_k-$F\mathsf{L}^{\tilde{\Sigma}_{i-1k}^{\mathsf{b}}}[wit]$.

Lemma 4 *TLS_k^1 proves there are $\tilde{\Sigma}_{1,k}^{\mathsf{b},\{|id|\}}$, L_k-FL function definitions of the identity function and each of the L_k-base functions such that the output of these functions on a given input match what the corresponding function would output.*

Proof. As most are relatively straightforward, we sketch the idea for a couple of them. Let $L_f = \{\langle i, x\rangle | \mathrm{BIT}(i, x) = 1\}$. To compute $\mathrm{BIT}(i, x) = 1$ in L, on input i and x, first copy i to an auxiliary tape, then on another tape count from 0 in binary to i while moving along x. After reaching i, query this position of x and check if it is 1. Such a computation and final configuration of a machine doing this is expressible by a $\mathsf{D\bar{d}H}_{1,1}^{\{p(|id|)\}}$-$ITER$ axiom and the correctness of what is computed by this machine can be checked by open-$LIND$. We can set $t_f := x$ and then $i \leq |t_f|$, $\mathrm{BIT}(i, y) = 1 \Leftrightarrow \langle i, x\rangle \in L_f$ shows the identity function is L_k-FL defined. Probably the hardest base functions are '+' and '$\cdot$'. The grade school algorithms to compute whether $\mathrm{BIT}(i, x + y) = 1$ or $\mathrm{BIT}(i, x \cdot y) = 1$ are in L, so for these again, use $\mathsf{D\bar{d}H}_{1,1}^{\{p(|id|)\}}$-$ITER$ to show the existence of the computation sequences of their corresponding machines, followed by a finite number of open-$LIND$ arguments on configuration sequences to argue the correctness. □

Lemma 5 *For $i \geq 1, k = 1, 2$, TLS_k^i (resp. TSC_k^i) and for $i > 1$, $\breve{S}_k^{i-1}$ proves L_k-$F\mathsf{L}^{\tilde{\Sigma}_{i-1,k}^{\mathsf{b}}}[wit]$ (resp.L_k-$F\mathsf{SC}^{\tilde{\Sigma}_{i-1,k}^{\mathsf{b}}}[wit]$) are closed under composition.*

Proof. The TSC_k^i and the TLS_k^i result is proven similarly, so we show only the latter. Let T denote TLS_k^i or $\breve{S}_k^{i-1}$. Suppose f and g are $\breve{\Sigma}_i^{\mathsf{b},\{p(|id|)\}}$ defined in $\breve{S}_k^{i-1}$ via formulas $A_g(y, \vec{y'}, z)$ and $A_f(\vec{x}, y)$. Then $\breve{S}_k^{i-1}$ proves

$$\exists y \leq t(\vec{x}) A_f(\vec{x}, y) \wedge \exists w \leq t'(y, \vec{y'}) A_g(y, \vec{y'}, w)$$

where the bounds t and t' are provable by Parikh's theorem, so $\breve{S}_k^{i-1}$ can $\breve{\Sigma}_i^{\mathsf{b},\{p(|id|)\}}$-define the composition $g(f(\vec{x}), \vec{y'}) = z$ of f and g. This definition though is not expressed as an L_k-$F\mathsf{L}^{\tilde{\Sigma}_{i,k}^{\mathsf{b}}}[wit]$ computation. To show the latter, let M_f and M_g be check whether $\langle i, \vec{x}, \vec{z}\rangle \in L_f$

and whether $\langle j, y, \vec{y'}, \vec{z'}\rangle \in L_g$ respectively. Let $|t_f|$ and $|t_g|$ be terms bounding the length of the outputs. We now build $M_{g\circ f}$ to check whether $\langle j, \vec{x}, \vec{y'}, \vec{z}, \vec{z'}\rangle \in L_{g\circ f}$. To initialize $M_{g\circ f}$'s on $\langle j, \vec{x}, \vec{y'}, \vec{z}, \vec{z'}\rangle$, $M_{g\circ f}$ on new tapes determines the lengths of j, $\vec{x}$, $\vec{y'}$, $\vec{z}$. It then computes the length of $y = f(\vec{x})$ for the witness choices $\vec{z}$, by checking memberships of $\langle i, \vec{x}, \vec{z}\rangle \in L_f$ using M_f for each value $i \leq |t_f|$, reusing space. As M_f is computed in logspace, this will also be logspace. Then $M_{g\circ f}$ operates by simulating M_g. If M_g is about to enter the input query state with t (we assume t is written on a tape for the simulation not $M_{g\circ f}$'s first work tape) written on its first work tape, $M_{g\circ f}$ computes if the input had been $\langle j, y, \vec{y'}, \vec{z'}\rangle$ which position of j, y, $\vec{y'}$, $\vec{z}'$ would have been queried. For t in the ranges of j, $\vec{y'}$, $\vec{z'}$, an appropriate modified t' is computed on the first work tape and it makes a query of $\langle j, \vec{x}, \vec{y'}, \vec{z}, \vec{z'}\rangle$. If t is in the range of positions of y, then $M_{g\circ f}$ simulates M_f on appropriate t'' in $\langle t'', \vec{x}, \vec{z}\rangle$ (where symbols in $\vec{x}$ and $\vec{z}$ have to be calculated as well from the original input). The total space space used for this would be proportional to the sum of the space used by f and g and so would be logspace and in this way $M_{g\circ f}$ could recognize $L_{g\circ f}$. Let $A_{g\circ f}(\vec{x}, \vec{y'}, w)$ be the $\tilde{\Sigma}^{\mathsf{b},\{p(|id|)\}}_i$ defining formula given by Corollary 1 that uses the appropriate witness queries for the $\vec{z}$ and $\vec{z'}$ variables and then computes its output via bit comprehension and membership checking $L_{g\circ f}$ as computed by $M_{g\circ f}$ as per formula (1). TLS^i_k and $\breve{S}^{i-1}_k$ could then show

$$\exists y \leq t(\vec{x}) A_f(\vec{x}, y) \wedge y \leq t(\vec{x}) \wedge w \leq t'(y, \vec{y'}) \wedge A_g(y, \vec{y'}, w) \Leftrightarrow$$
$$w \leq t'(t(\vec{x}), \vec{y'}) \wedge A_{g\circ f}(\vec{x}, \vec{y'}, w)$$

arguing based on the transition function of f that when $M_{g\circ f}$ uses a bit from the input parameter associated with f when simulating g, it would correspond to a bit of a y that satisfies $A_f(\vec{x}, y)$. □

Lemma 6 *For $i \geq 1$, let $A \in \mathsf{D\bar{d}H}^{\tau}_{i+1}$ and $B \in \mathsf{L}\tilde{\Sigma}^{\mathsf{b}}_i$. If $\tau := \{p(|id|)\}$ (resp. $\tau := \{2^{p(||id||)}\}$), then $\breve{S}^i_k \subseteq TLS^{i+1}_k$ (resp. TSC^i_k) proves the graph of A and the function $\mu j < |t| A(j, \vec{a})$ are in L_k-$F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}$ using a $\tilde{\Sigma}^{\mathsf{b},\tau}_{i+1}$-definition. They also show $(Wj \leq t)B(j, \vec{a})$ is in L_k-$F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}[wit]$ (resp. L_k-$F\mathsf{SC}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}[wit]$) using a $\tilde{\Sigma}^{\mathsf{b},\tau}_{i+1}$-definition. For $i = 0$, the graph of A and μ-operator results hold of TLS^1_k (resp. TSC^1_k) and one can restrict the result respectively to $F\mathsf{L}$ or $F\mathsf{SC}$.*

Proof. We show the $i > 0$, $\breve{S}^i_k$ case, but the $i = 0$ and TSC^i_k cases are similar. That $\breve{S}^i_k$ proves the graph of $A \in \mathsf{D\bar{d}H}^{\{p(|id|)\}}_{i+1}$ is in L_k-$F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}$ is proven by induction on the complexity of the formula A. In the base case, suppose $A(\vec{x}) \in B(\tilde{\Sigma}^{\mathsf{b}}_{i,k})$. In this case, rewrite A in conjunctive normal form. The graph of A will be in L_k-$F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}$ if $L_A = \{\langle 0, \vec{x}\rangle | A(\vec{x})\}$ is in $\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}$. The 0 is for the 0th bit of the 0-1 valued graph of A. L_A is recognized by a finite step, finite space machine M_A that cycles over each disjunct conjuncted together in A and for each such disjunct makes finitely many $\tilde{\Sigma}^{\mathsf{b}}_{i,k}$ oracle queries to see if any of its atoms is true. So the graph of A is $\tilde{\Sigma}^{\mathsf{b},\{p(|id|)\}}_{i+1}$ definable in $\breve{S}^i_k$ by Corollary 1 via some formula B_A. That $A(\vec{x}) \Leftrightarrow B_A(0, \vec{x}, 1)$ holds (we view the parameters $0, \vec{x}$, as the inputs and 1 is the output) could be checked by cases as the computation of M_A is finite. Now consider (ℓ, ϵ)-iteration formula $\mathrm{Iter}_{t_1,t_2,B_1}(C, C', d, \vec{a})$ where $\ell \in \{p(|id|)\}$, t_1 is an L_k-term, and $B_1(c, c', \vec{a})$ has already been given an L_k-$F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}$ algorithm. We assume also by induction that $\breve{S}^i_k \vdash \exists! c' \leq \ell(t_1) B_1(c, c', \vec{a})$. Let M_{B_1} be a $\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}$ machine for the graph of B_1. Since $\ell(x) \leq |x|^m$ for some m. Starting from 0, counting up, reusing space, cycling over the possible values of c', then computing M_{B_1} on that c', a $\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}$ machine, $M_{B'}$, could obtain a c' such that $B_1(c, c', \vec{a})$ holds. Further, since $\breve{S}^i_k$ proves the uniqueness of c', it proves that the value obtained by $M_{B'}$ matches the c' such that $B_1(c, c', \vec{a})$. Starting at C and running $M_{B'}$ reusing space $\min(d, \ell^{\epsilon})$ times using c' as the c input for the following time, one obtains a $\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}$ machine for $\mathrm{Iter}_{t_1,t_2,B_1}(C, C', d, \vec{a})$. Correctness of this algorithm in $\breve{S}^i_k$ can first be proven using $\breve{\Sigma}^{\mathsf{b},\{|id|\}}_{i,k}$-*LIND* for the algorithm run where queries are only answered according to an arbitrary query string and positive answers are correct with respect to the $\tilde{\Sigma}^{\mathsf{b}}_{i,k}$ oracle, and then extended to a maximal such query string. Using Lemma 5 and Lemma 4, one can handle (ℓ, ϵ)-iteration formula with accept states.

Suppose we need to compute $\mu j < |t| A(j, \vec{a})$ where $A(c, \vec{a}) \in \mathsf{D\bar{d}H}^{\{p(|id|)\}}_{i+1}$. By the previous result, A's graph is computed by a $\tilde{\Sigma}^{\mathsf{b},\{p(|id|)\}}_{i+1}$-definable f_A in L_k-$F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}$. On input $\langle i, \vec{a}, \rangle$ to see if the ith bit of $\mu j < |t| A(j, \vec{a})$ is '1', using the machine M_A that computes the language L_A used to define f_A, one can compute $A(j, \vec{a})$ for successive values j until either $j = |t| + 1$ or we determine $A(j, \vec{a})$ holds. For this j we accept only if the ith bit is 1. This computation will be in $\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}$ and correctness properties can be proven in a similar fashion to the previous result.

For the W-operator result, let $f := (Wj \leq t)B(j, \vec{a})$ for $B \in \mathsf{L}\tilde{\Sigma}^{\mathsf{b}}_i$. Define $A^f_1 := B$, $s_1 = t$, and the language $L_f = \{\langle i, x, z\rangle | \mathrm{BIT}(i, z) = 1\}$. Then the graph of L_f is in L_k-$F\mathsf{L}$ via Lemma 4 and Lemma 5 and A^f_1, s_1, L_f define $f \in L_k$-$F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,k}}[wit]$ computing $f := (Wj \leq t)B(j, \vec{a})$, so it is $\tilde{\Sigma}^{\mathsf{b}}_{i+1}$-defined in $\breve{S}^i_k$. □

4 Witnessing

We prove the converse to Theorem 1 using a witnessing argument. Given $A \in \mathsf{LE}\tilde{\Sigma}^{\mathsf{b},\tau}_i$, define a term t_A and a formula $WIT^{i,\tau}_A$ as follows:

- If $A(\vec{a}) \in \mathsf{LD\bar{d}H}^{\tau}_i$ then $t_A := 0$ and $WIT^{i,\tau}_A(w, \vec{a}) := w = 0 \wedge A(\vec{a})$.
- If $A(\vec{a}) \in \mathsf{LE}\tilde{\Sigma}^{\mathsf{b},\tau}_i \setminus \mathsf{LD\bar{d}H}^{\tau}_i$ is of the form $\exists x \leq tB(x, \vec{a})$, then $t_A := 4 \cdot (2^{2|\max(t, t_B)|})$ and

$$WIT^{i,\tau}_A(w, \vec{a}) := \mathrm{ispair}(w) \wedge (w)_1 \leq t \wedge WIT^{i,\tau}_B((w)_2, (w)_1, \vec{a}),$$

Given a cedent of formulas Γ, write $\bigwedge\!\!\!\bigwedge \Gamma$ for their conjunction, $\bigvee\!\!\!\bigvee \Gamma$ for their disjunction, and extend the definition of witness to such cedents iteratively by defining the witness to an empty antecedent to be the formula $w = w$, the witness for an empty succedent to be $\neg w = w$, and define a witness for: $A \wedge \bigwedge\!\!\!\bigwedge \Gamma$, $A \vee \bigvee\!\!\!\bigvee \Gamma$, by setting $t_{A \wedge \bigwedge\!\!\!\bigwedge \Gamma} := 4 \cdot (2^{2|\max(t_B, t_C)|})$, $t_{A \vee \bigvee\!\!\!\bigvee \Gamma} := 4 \cdot (2^{2|\max(t_B, t_C)|})$ and defining

$$WIT^{i,\tau}_{A \wedge \bigwedge\!\!\!\bigwedge \Gamma}(w, \vec{a}) \quad := \quad \mathrm{ispair}(w) \wedge WIT^{i,\tau}_A((w)_1, \vec{a}) \wedge \; WIT^{i,\tau}_{\bigwedge\!\!\!\bigwedge \Gamma}((w)_2, \vec{a}) \qquad (2)$$

$$WIT^{i,\tau}_{A \vee \bigvee\!\!\!\bigvee \Gamma}(w, \vec{a}) \quad := \quad \mathrm{ispair}(w) \wedge (\, WIT^{i,\tau}_A((w)_1, \vec{a}) \vee \; WIT^{i,\tau}_{\bigvee\!\!\!\bigvee \Gamma}((w)_2, \vec{a})), \qquad (3)$$

The following lemma is true for the witness predicate:

Lemma 7 *If $A(\vec{a}) \in \tilde{\Sigma}^{\mathsf{b},\tau}_i$, Γ a cedent of $\tilde{\Sigma}^{\mathsf{b}}_i$ formulas, then: (a) For $i > 0$, $WIT^{i,\tau}_A$, $WIT^{i,\tau}_{\bigwedge\!\!\!\bigwedge \Gamma}$, and $WIT^{i,\tau}_{\bigvee\!\!\!\bigvee \Gamma}$ are logically equivalent to $\mathsf{D\bar{d}H}^{\tau}_i$-predicates. (b) For $i > 0$, $BASIC^1_k \vdash \exists w \leq t_A(\vec{a})\, WIT^{i,\tau}_A(w, \vec{a}) \Leftrightarrow A(\vec{a})$.*

Proof. Part (a) follows from the definition of witness and since $\hat{\beta}$ and the pairing functions are defined by L_1-terms. Part (b) is easily proved by induction on the complexity of A. □

Theorem 2 *Let $TL_{i,k}$ and $FL_{i,k}$ be TLS^1_k and L_k-$F\mathsf{L}$, if $i = 1$, and $\breve{S}^{i-1}_k$ and L_k-$F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i-1,k}}[wit]$ if $i > 1$. Let $FS_{i,k}$ be L_k-$F\mathsf{SC}$, if $i = 1$, and be L_k-$F\mathsf{SC}^{\tilde{\Sigma}^{\mathsf{b}}_{i-1,k}}[wit]$ if $i > 1$.*

(a) For $i > 1, k = 1, 2$, if $TLS^i_k \vdash \Gamma \to \Delta$ where Γ and Δ are cedents of $\mathsf{LE}\tilde{\Sigma}^{\mathsf{b},\{p(|id|)\}}_i$ formulas, then there is an $f \in FL_{i,k}$, $\tilde{\Sigma}^{\mathsf{b},\{p(|id|)\}}_i$ definable in $TL_{i,k}$ via formula A_f and $TL_{i,k} \vdash A_f(w, \vec{a}, z) \wedge WIT^{i,\{|id|\}}_{\bigwedge\!\!\!\bigwedge \Gamma}(w, \vec{a}) \supset WIT^{i,\{p(|id|)\}}_{\bigvee\!\!\!\bigvee \Delta}(z, \vec{a})$.

(b) For $i \geq 1, k = 1, 2$, but for the $i = 1$ case without an witness oracle, if $TSC^i_k \vdash \Gamma \to \Delta$ where Γ and Δ are cedents of $\mathsf{LE}\tilde{\Sigma}^{\mathsf{b},\{2^{p(||id||)}\}}_i$ formulas, then there is a $f \in FS_{i,k}$, $\tilde{\Sigma}^{\mathsf{b},\{2^{p(||id||)}\}}_i$ definable in TSC^i_k via formula A_f and $TSC^i_k \vdash A_f(w, \vec{a}, z) \wedge WIT^{i,\{2^{p(||id||)}\}}_{\bigwedge\!\!\!\bigwedge \Gamma}(w, \vec{a}) \supset WIT^{i,\{2^{p(||id||)}\}}_{\bigvee\!\!\!\bigvee \Delta}(z, \vec{a})$.

Proof. The proofs of (b) and the $i = 1$ case of (a) are similar to the $i > 1$ case of (a), which is harder. All of these rely on their respective sub-cases of Lemma 5. So we show only (a) for $i > 1$. The proof of (a) is by induction on the number of sequents in a TLS^i_1 proof of $\Gamma \to \Delta$. By cut elimination, all the sequents in the proof are $\mathsf{LE}\tilde{\Sigma}^{\mathsf{b},\{p(|id|)\}}_i$. The base cases involves open initial sequents, *BASIC* axioms, $open_k$-*LIND* axioms, or $\mathsf{D\bar{d}H}^{\{p(|id|)\}}_i$-*ITER* axioms which are each witnessed by 0, so trivial. For the induction step, the proof splits into cases according to the last inferences in the TLS^i_1 proof. We show below the cases which are different from previous witness proofs and invite the reader to consult Krajíček [8] for cases previously considered in the bounded arithmetic literature.

($\exists$:right case) Suppose we have the inference:

$$\frac{\Gamma \to A(t), \Delta}{t \le s, \Gamma \to (\exists x \le s)A(x), \Delta}$$

By hypothesis, there is a $g \in L_k\text{-}F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i-1,k}}[wit]$ such that

$$\breve{S}^{i-1}_k \vdash A_g(w, \vec{a}, z) \wedge WIT^i_{\bigwedge\!\!\!\bigwedge\Gamma}(w, \vec{a}) \supset WIT^i_{A(t)\vee(\bigvee\!\!\!\bigvee\Delta)}(z, \vec{a}).$$

The definition of $WIT^{i,\{p(|id|)\}}_{t\le s\wedge(\bigwedge\!\!\!\bigwedge\Gamma)}(w, \vec{a})$ implies

$$\breve{S}^{i-1}_k \vdash WIT^{i,\{p(|id|)\}}_{t\le s\wedge(\bigwedge\!\!\!\bigwedge\Gamma)}(w, \vec{a}) \supset t \le s \wedge WIT^{i,\{p(|id|)\}}_{\bigwedge\!\!\!\bigwedge\Gamma}(w, \vec{a})$$

If $(\exists x \le s)A(x) \in \mathsf{LE}\tilde{\Sigma}^{\mathsf{b},\{p(|id|)\}}_i \setminus \mathsf{LD\bar{d}H}^{\{p(|id|)\}}_i$, define $f := \langle t(\vec{a}), (g((w)_2, a))_2\rangle$. Otherwise, define $f := g((w)_2, a)$. These function are definable in $\breve{S}^{i-1}_k$ by Lemma 5 and using properties of the pairing function provable in $\breve{S}^{i-1}_k$, $\breve{S}^{i-1}_k$ proves

$$A_f(w, \vec{a}, z) \wedge WIT^{i,\{p(|id|)\}}_{t\le s\wedge(\bigwedge\!\!\!\bigwedge\Gamma)}(w, \vec{a}) \supset WIT^{i,\{p(|id|)\}}_{(\exists x\le s)A(x)\vee(\bigvee\!\!\!\bigvee\Delta)}(z, \vec{a}).$$

($\forall$:right case) Suppose we have the inference:

$$\frac{b \le t, \Gamma \to A(b), \Delta}{\Gamma \to (\forall x \le t)A(x), \Delta}$$

By hypothesis, there is a $g \in L_k\text{-}F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i-1,k}}[wit]$ such that

$$\breve{S}^{i-1}_k \vdash A_g(w, \vec{a}, b, z) \wedge WIT^{i,\{p(|id|)\}}_{b\le t\wedge(\bigwedge\!\!\!\bigwedge\Gamma)}(w, b, \vec{a}) \supset WIT^{i,\{p(|id|)\}}_{A(b)\vee(\bigvee\!\!\!\bigvee\Delta)}(z, b, \vec{a}).$$

If $(\forall x \le t)$ is bounded but not sharply bounded, then $A \in \mathsf{L}\breve{\Pi}^{\mathsf{b}}_{i-1}$ because for formulas in $\mathsf{D\bar{d}H}^{\{p(|id|)\}}_i = \mathsf{D\bar{d}H}^{\{p(|id|)\}}(B(\breve{\Sigma}^{\mathsf{b}}_{i-1}))$, the outer universal quantifiers not coming from the $B(\breve{\Sigma}^{\mathsf{b}}_{i-1})$ subformula are all sharply bounded. So there are two cases to consider: Where $A \in \mathsf{L}\breve{\Pi}^{\mathsf{b}}_{i-1}$ and where $A \in \mathsf{LD\bar{d}H}^{\{p(|id|)\}}_i \setminus \mathsf{L}\breve{\Pi}^{\mathsf{b}}_{i-1}$.

First, suppose $A \in \mathsf{L}\breve{\Pi}^{\mathsf{b}}_{i-1}$, by Lemma 5, $(Wx \le t)\neg A$ is in $L_k\text{-}F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i-1,k}}[wit]$. For the value that this multifunction returns, $\breve{S}^{i-1}_k$ either proves $(\forall x \le t)A(x)$ is valid or the second component of the witness returned by g must witness Δ. So $\breve{S}^{i-1}_k$ can $\tilde{\Sigma}^{\mathsf{b}}_i$-define the multifunction $h(w, \vec{a}) := \langle 0, (g(w, \vec{a}, (Wx \le t)\neg A))_2\rangle$ and show it witnesses the lower sequent.

Otherwise, suppose $A \in \mathsf{LD\bar{d}H}^{\{p(|id|)\}}_i \setminus \mathsf{L}\breve{\Pi}^{\mathsf{b}}_{i-1}$. By cut-elimination, $(\forall x \le t)A(x)$ must then match the lexical shape of some $\mathsf{D\bar{d}H}^{\{p(|id|)\}}_i$ formula. So t must be of the form $t = \ell^{\epsilon}(s)$ for some $0 < \epsilon \le 1$. From the witness predicate definition: $WIT^{i,\{p(|id|)\}}_{(\forall x\le t)A(x)\vee(\bigvee\!\!\!\bigvee\Delta)}(z, \vec{a})$ is

$$\text{ispair}(z) \wedge (WIT^{i,\{p(|id|)\}}_{(\forall x\le \ell^{\epsilon}(s))A(x)}((z)_1, \vec{a}) \vee WIT^{i,\{p(|id|)\}}_{\bigvee\!\!\!\bigvee\Delta}((z)_2, \vec{a})).$$

So

$$\breve{S}^{i-1}_k \vdash A_g(w,\vec{a},b,z) \wedge WIT^{i,\{p(|id|)\}}_{b\le t\wedge(\wedge\Gamma)}(w,b,\vec{a}) \wedge \mathrm{ispair}(z) \wedge$$
$$\neg WIT^{i,\{p(|id|)\}}_{(\vee\Delta)}((z)_2,\vec{a}) \supset (b \le t \supset WIT^{i}_{A(b)}((z)_1,b,\vec{a}))$$

We note $\breve{S}^{i-1}_k$ proves $WIT^{i,\{p(|id|)\}}_{A(b)}((z)_1,b,\vec{a})$ implies $\exists z' \le t_A\, WIT^{i,\{p(|id|)\}}_{A(b)}(z',b,\vec{a})$ Since b is an eigenvariable, this shows $\breve{S}^{i-1}_k$ proves $b \le t \supset WIT^{i,\{p(|id|)\}}_{A(b)}((z)_1,b,\vec{a})$ implies $\forall x \le \ell^\epsilon(s)\exists z' \le t_A\, WIT^{i}_{A(x)}(z',x,\vec{a})$ which is $WIT^{i}_{(\forall x\le \ell^\epsilon(s))A(x)}$ except for the condition $w=0$. From this it follows that the $\breve{S}^{i}_k$ proves the multifunction

$$\langle 0, (g(w,\vec{a},(\mu b \le \ell^\epsilon(s))\neg WIT^{i,\{p(|id|)\}}_{A(b)}((g(w,\vec{a},b))_1,\vec{a},b)))_2\rangle$$

witnesses either $WIT^{i,\{p(|id|)\}}_{\mathbb{W}\Delta}$ or $WIT^{i,\{p(|id|)\}}_{(\forall x\le \ell^\epsilon(s))A(x)}$. By Lemma 5, this multifunction is in L_k-$F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i-1,k}}[wit]$.
□

Corollary 2 *The following statements hold:*

(a) For $i>1, k=1,2$, the $\tilde{\Sigma}^{\mathsf{b},\{p(|id|)\}}_i$-defined multifunctions of TLS^i_k and $\breve{S}^{i-1}_k$ are exactly $F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i-1,k}}[wit]$. The $\tilde{\Delta}^{b,\{|id|\}}_i$-predicates are exactly $\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i-1,k}}$.

(b) For $k=1,2$, the $\tilde{\Sigma}^{\mathsf{b}}_1$-defined functions of TLS^1_k are exactly $F\mathsf{L}$ and the $\tilde{\Delta}^{b,\{|id|\}}_1$-predicates are exactly L.

(c) The $\tilde{\Sigma}^{\mathsf{b},\{2^{p(||id||)}\}}_i$-defined multifunctions of TSC^i_1 are exactly in $F\mathsf{SC}^{\tilde{\Sigma}^{\mathsf{b}}_{i-1,k}}[wit]$ and the $\tilde{\Delta}^{b,\{2^{p(|id|)}\}}_i$-predicates are exactly $\mathsf{SC}^{\tilde{\Sigma}^{\mathsf{b}}_{i-1,k}}$.

(d) For $k=1,2$, the $\tilde{\Sigma}^{\mathsf{b}}_1$-defined functions of TSC^1_k are exactly $F\mathsf{SC}$ and the $\tilde{\Delta}^{b,\{2^{p(|id|)}\}}_1$-predicates are exactly SC.

Proof. Each of these is proven in the same way, invoking the appropriate sub-case of Theorem 2. We show only (a). By Theorem 1, $\breve{S}^{i-1}_k$ can $\tilde{\Sigma}^{\mathsf{b},\{p(|id|)\}}_i$ define all the multifunctions in $F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i-1,k}}[wit]$. For the other direction, suppose $TLS^i_k \vdash \forall x\exists y A(x,y)$. By Parikh's Theorem, $TLS^i_k \vdash (\exists y \le t)A(x,y)$ for some term t. Taking Γ to be empty and Δ to be the $\mathsf{E}\tilde{\Sigma}^{\mathsf{b},\{p(|id|)\}}_{i,k}$-formula $(\exists y\le t)A(x,y)$ in Theorem 2, we get $\breve{S}^i_k$ proves that there is a $\tilde{\Sigma}^{\mathsf{b},\{p(|id|)\}}_{i,k}$-definable, $F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i-1,k}}[wit]$ multifunction f with $\tilde{\Sigma}^{\mathsf{b}}_i$-formula graph A_f such that: $A_f(w,\vec{a},z) \supset WIT^{i,\{p(|id|)\}}_{(\exists y\le t)A(x,y)}((z)_1,\vec{a})$. The definition of $WIT^{i,\{p(|id|)\}}$ entails $\breve{S}^i_k$ proves $A_f(w,\vec{a},z) \supset A(x,((z)_1)_1)$, so $\breve{S}^i_k$ can find at least one value y such that $A(x,y)$ holds. Let k compute $((f)_1)_1$. Suppose $A(x,y)$ is of the form $(\exists z\le t)B(x,y,z)$, where $B \in \mathsf{D\bar{d}H}^{\{p(|id|)\}}_i$. Let f' be the multifunction that: (a) Compute $k(x)=y_0$. (b) Ask the queries $(Wy\le t)(y=y)$ and $(Wz\le s)(z=z)$. Let y_1 and z_1 be the oracle responses. (c) Compute the $\mathsf{D\bar{d}H}^{\{p(|id|)\}}_i$ predicate $\neg B(x,y1,z1)$. If the answer is '1' output y_0. Otherwise, output y_1. For $i>1$, f can be constructed using cond, Lemma 6, and Lemma 5, we have $f' \in F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i-1,k}}[wit]$. The purpose of step (b) is to nondeterministically get values for y_1 and z_1. If these values happen to witness $(\exists y\le t)A$ then y_1 is output, otherwise y_0 is output. For the (b) and (d) cases, since the object being defined is a function rather than a multifunction, there is a unique y satisfying $A(x,y)$. So we have the $\tilde{\Sigma}^{\mathsf{b}}_1$-defined functions of TLS^1_k are precisely $F\mathsf{L}$ and those of TSC^1_k are precisely $F\mathsf{SC}$.

From the above argument for the first part of (a) to show the $\tilde{\Delta}^{b,\{p(|id|)\}}_i$-predicates are precisely $\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i-1,k}}$, let $B(x)$ be $\tilde{\Delta}^{b,\{p(|id|)\}}_i$ in TLS^i_k. Then $TLS^i_k \vdash B(x) \Leftrightarrow B^\Sigma \Leftrightarrow B^\Pi$ for some $B^\Sigma \in \tilde{\Sigma}^{\mathsf{b},\{p(|id|)\}}_i$ and $B^\Pi \in \tilde{\Pi}^{\mathsf{b},\{p(|id|)\}}_i$. So the formula $B'(x,y) := (y=0 \wedge \neg B^\Pi(x)) \vee (y=1 \wedge B^\Sigma(x))$ is provably equivalent to a $\tilde{\Sigma}^{\mathsf{b},\{p(|id|)\}}_{i,k}$ formula in $\breve{S}^{i-1}_k$. Moreover, by consistency and excluded middle, $TLS^i_k \vdash \forall x \exists! y \leq 1 B'(x,y)$. Hence, by the theorem $\breve{S}^{i-1}_k$ proves this function can be witnessed by a multifunction in $f \in F\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i-1,k}}[wit]$. Since there is only one value of y that will witness $B'(x,y)$ for a given x, f must in fact be a function and will be 0 and 1 valued. Let $L_f \in \mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i-1,k}}$ be f defining language and suppose it uses $A^f_j \in \tilde{\Sigma}^{\mathsf{b}}_{i-1,k}$ for the witness query to compute z_j. So $B(x)$ holds iff there are witnesses from the A^f_j's such that $\langle 0, x, \vec{z}\rangle \in L_f$. Let M^A be a $\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i-1,k}}$ machine for L_f. Let A' be $\tilde{\Sigma}^{\mathsf{b}}_{i-1,k}$ oracle which outputs 'yes' if it can guess a $\vec{z}$ satisfying the $A^f_1, ..., A^f_r$ and a computation of M on $\langle 1, x, \vec{z}\rangle$ where the first m queries are answered according to a string q and if $\mathrm{BIT}(m', q) = 1$ for $m' \leq m$, then for the m'th query, q_m, $A(q_{m'})$ holds. Given M one can build a $\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i-1,k}}$ machine M' for $B(x)$ as follows: M' binary searches over longer and longer q's to find a query response string q with a maximal number of 'yes' answers. For this string the 'no' answers must also be correct. It then asks one more oracle q and $\tilde{\Sigma}^{\mathsf{b}}_{i-1,k}$ oracle A'' which computes as A' but also checks if the computation was accepting. $\Box$

Corollary 3 *For $i \geq 1, k \geq 2$, $S^i_k \preceq_{\forall B(\tilde{\Sigma}^{\mathsf{b}}_{i+1,k})} TLS^i_k$.*

Proof. If TLS^i_k proves a sequent $\rightarrow A(\vec{a})$ where A in the former cases is in $\tilde{\Sigma}^{\mathsf{b}}_1$ or in the latter case $\hat{\Sigma}^{\mathsf{b}}_{i+1,k}$, then by witnessing S^i_k proves $A_f(0, \vec{a}, v) \rightarrow WIT^i_A(v, \vec{a})$ for some $\tilde{\Sigma}^{\mathsf{b}}_i$-definable multifunction f. Here the 0 is the witness for the empty precedent. So Lemma 7 entails $A_f(0, \vec{a}, v) \rightarrow A(\vec{a})$. As f is definable, S^i_k proves $\exists v A_f(0, \vec{a}, v)$ and so after an exists left rule and a cut, S^i_k proves $A(\vec{a})$. The universal closure of the provability of such formulas shows $\forall\tilde{\Sigma}^{\mathsf{b}}_i$ conservativity.

To show conservativity for Boolean formulas (and hence also their universal closure), suppose T is either TLS or TLS^i_k proves a sequent $\rightarrow A(\vec{a})$ where A is a Boolean combination of Σ-formulas where Σ is respectively $\hat{\Sigma}^{\mathsf{b}}_1$ or $\hat{\Sigma}^{\mathsf{b}}_{i+1,k}$. So A is equivalent to a formula of the form $\wedge_n \vee_j A_{nj}$ where A_{nj} is either a Σ formula or its negation. So T proves each conjunct, and each conjunct can be rewritten a sequent $\Sigma \rightarrow \Delta$ of Σ formulas and then conservativity again follows using Theorem 2. $\Box$

5 Independence

In this section, we prove independence results for TSC^1_1 and $I\Delta_0$ that follow from our definability results.

Lemma 8 *There is $\hat{\Sigma}^{\mathsf{b}}_{i,1}$-formula $U_i(e, x, z)$ such that for any $\hat{\Sigma}^{\mathsf{b}}_{i,2}$-formula $A(x)$ there is a number e_A and L_2-term t_A for which $TLS^1_2 \vdash U_i(e_A, x, t_A(x)) \equiv A(x)$. If A is in $\hat{\Sigma}^{\mathsf{b}}_{i,1}$ then t_A can be chosen to be an L_1-term in x or we can choose a single L_2-term $t(e_A, x)$ which works for all A.*

Proof. This is shown for TLS in Pollett [14]. The same argument holds in TLS^1_2 since it only involves finite manipulation of sequences needed to compute A on x based on its Gödel coding. We include it for completeness.

Using $\mathrm{K}_\neg(x) := 1 \dot{-} x$, $\mathrm{K}_\vee(x,y) := x + y$, and $\mathrm{K}_\leq(x,y) := \mathrm{K}_\neg(y \dot{-} x)$, any open formula $A(x, \vec{y})$ is equivalent to an equation $f(x, \vec{y}) = 0$ where $f \in L_k$. By induction on the complexity of A, this is provable in TLS^1_2. This entails any $\hat{\Sigma}^{\mathsf{b}}_i$-formula $\phi(x)$ is provably equivalent in TLS^1_2 to one of the form

$$(\exists y_1 \leq t_1) \cdots (Q y_i \leq t_i)(Q' y_{i+1} \leq |t_{i+1}|)(t_{i+2}(x, \vec{y}) = 0)$$

where the quantifiers Q and Q' will depend on whether i is even or odd. Fix some coding scheme for the 12 symbols of L_2 and the $i+2$ variables $x, y_1, \ldots, y_{i+1}$. We use $\ulcorner\urcorner$ to denote the code for some symbol. i.e., $\ulcorner = \urcorner$ is the code for $=$. We choose our coding so that all codes require less than $|i + 14|$ bits and 0 is used as $\ulcorner NOP \urcorner$ meaning no operation. Thus, if one tries to project out operations beyond the end of the code of the term one naturally just projects out $\ulcorner NOP \urcorner$'s. The code for a term t is a sequence of blocks of length $|i+14|$ that write out t in postfix order. So $x + y_1$ would be coded as the three blocks $\ulcorner x \urcorner \ulcorner y_1 \urcorner \ulcorner + \urcorner$. The code for a $\hat{\Sigma}^{\mathsf{b}}_i$-formula will be $\langle\langle \ulcorner t_1 \urcorner, \ldots, \ulcorner t_{i+3} \urcorner \rangle\rangle$. Given this we obtain $U_i(e, x, z)$ from the formula

$$(\exists w \leq z)(\exists y_1 \leq z)(\forall j \leq |e|)(\forall y_2 \leq z) \cdots$$
$$\cdots (Q y_i \leq z)(Q' y_{i+1} \leq |z|)\phi_i(e, j, x, \vec{y})$$

after pairing is applied. Here ϕ_i consists of a statement saying w is a tuple of the form $\langle\langle w_1, \ldots, w_{i+2} \rangle\rangle$ together with statements saying each w_i codes a postfix computation of t_i in $e = \langle\langle \ulcorner t_1 \urcorner, \ldots, \ulcorner t_{i+3} \urcorner \rangle\rangle$. If $z' := MSP(z, \lfloor \frac{1}{2}|z| \rfloor)$ (roughly, the square root of z) is used as the block size, this amounts to checking conditions for each m

$$[\hat{\beta}_{|i+14|}(j, \ulcorner t_m \urcorner) = \ulcorner x \urcorner \supset \hat{\beta}_{|z'|}(j, w_m) = x] \wedge$$
$$[\hat{\beta}_{|i+14|}(j, \ulcorner t_m \urcorner) = \ulcorner + \urcorner \supset$$
$$\hat{\beta}_{|z'|}(j, w_m) = \hat{\beta}_{|z'|}(j \dot{-} 2, w_m) + \hat{\beta}_{|z'|}(j \dot{-} 1, w_m)] \wedge \cdots$$
$$[\hat{\beta}_{|i+14|}(j, \ulcorner t_m \urcorner) = \ulcorner \# \urcorner \supset$$
$$|\hat{\beta}_{|z'|}(j, w_m)| = S(|\hat{\beta}_{|z'|}(j \dot{-} 2, w_m)||\hat{\beta}_{|z'|}(j \dot{-} 1, w_m)|)$$
$$\wedge\, LSP(\hat{\beta}_{|z'|}(j, w_m), |\hat{\beta}_{|z'|}(j, w_m)| \dot{-} 1) = 0] \wedge \cdots$$
$$\cdots$$
$$[\hat{\beta}_{|i+14|}(j, \ulcorner t_m \urcorner) = \ulcorner NOP \urcorner \supset \hat{\beta}_{|z'|}(j, w_m) = \hat{\beta}_{|z'|}(j \dot{-} 1, w_m)].$$

ϕ_i also has conditions $y_m \leq \hat{\beta}_{|z'|}(|e|, w_m) \wedge$ if y_m was existentially quantified and conditions $y_m \leq \hat{\beta}_{|z'|}(|e|, w_m) \supset$ if y_m was universally quantified. None of these conditions use the $\#$ function. Finally, ϕ_i has a condition saying $\hat{\beta}_{|z'|}(|e|, w_{i+2}) = 0$. Since TLS^1_2 proves simple facts about projections from pairs, it can prove by induction on the complexity of the terms in any $\hat{\Sigma}^{\mathsf{b}}_i$-formula $\phi(x)$ that $U_i(e_\phi, x, t(e_\phi, x)) \equiv \phi(x)$ provided $t(e_\phi, x)$ is large enough.

To estimate the size of t_A, an upper bound on w_m is calculated. First, all real formulas A have their terms represented as trees, so we can assume e_A codes terms which are trees. By induction over the subtrees of a given term t_m, one can show an upper bound on the block size needed to store a step of w_m of the form $|e_m|(|x| + |e_A|)$. So the length of any w_m can be bounded by $\ell = |e_A||e_A|(|x| + |e_A|) > |e_m||e_m|(|x| + |e_A|)$. So choosing an L_1-term larger than $2^{(i+2)\ell}$ suffices. This is possible since e_A is a fixed number. Notice if both e_A and x are viewed as parameters, this is in fact boundable by an L_2-term t. If A does involve $\#$ than a similar estimate can be done to show that an L_2-term for t_A suffices. $\square$

The above also holds for TSC^1_2 as $TLS^1_2 \subseteq TSC^1_2$.

Lemma 9 *For $i \geq 1$, $\hat{\Sigma}^{\mathsf{b}}_{i,1} \neq \hat{\Pi}^{\mathsf{b}}_{i,2}$. That is, there is a formula $\phi(x) \in \hat{\Pi}^{\mathsf{b}}_{i,2}$ such that for any $A(x)$ in $\hat{\Sigma}^{\mathsf{b}}_{i,1}$, $\mathbb{N} \not\models \forall x(\phi(x) \Leftrightarrow A(x))$.*

Proof. This result is from Pollett [14]. Again, we include the proof for completeness. If A is in $\hat{\Sigma}^{\mathsf{b}}_{i,1}$ then the last argument of U_i from Lemma 8 is an L_2-term. So there is a $\hat{\Sigma}^{\mathsf{b}}_{i,2}$-formula $U(x, e_A) \equiv A$ for all A in $\hat{\Sigma}^{\mathsf{b}}_{i,1}$. Consider $\neg U(x, x)$ this formula is equivalent to a $\hat{\Pi}^{\mathsf{b}}_{i,2}$-formula. Also, it is easy to see it is not in $\hat{\Sigma}^{\mathsf{b}}_{i,1}$. $\square$

Lemma 10 *For $i > 0$, let T be TLS^i_1 or TSC^i_1. If T proves the MRDP theorem then T proves $\mathsf{E}_1 = \mathsf{U}_1$.*

Proof. To see this suppose T proves the MRDP theorem. This would mean T could show for any formula $A \in \Sigma_1$ that it is equivalent to some formula $(\exists \vec{y}) P(\vec{x}, \vec{y}) = Q(\vec{x}, \vec{y})$ where P and Q are polynomials. In particular, as $\mathsf{U}_{1,k} \subseteq \Sigma_1$, for any $\mathsf{U}_{1,k}$-formula $A(\vec{x})$ there is a formula $F(\vec{x}) := (\exists \vec{y}) P(\vec{x}, \vec{y}) = Q(\vec{x}, \vec{y})$ where P, Q are polynomials such that $T \vdash A \equiv F$. This would mean T proves $A \to (\exists\vec{y})P(\vec{x},\vec{y}) = Q(\vec{x},\vec{y})$. By Parikh's theorem, since T is a bounded theory one can bound the $\vec{y}$'s by an L_k-term t giving an $\mathsf{E}_{1,k}$-formula F_2. Note $F_2 \supset F \supset A$ so $A \equiv F_2$ completing the proof. □

Theorem 3 *TSC_1^1 does not prove MRDP.*

Proof. By the previous lemma, if TSC_1^1 proves the MRDP Theorem then it proves any bounded formula, a Δ_0 formula, is equivalent to an E_1 formula and to a U_1 formula. So as $\tilde{\Sigma}^{\mathsf{b}}_{1,1}$, $\hat{\Pi}^{\mathsf{b}}_{1,1}$, $\hat{\Sigma}^{\mathsf{b}}_1$ each either contain E_1 or U_1, and are all bounded formulas, we have in terms of languages expressed by their constituent formulas that $\Delta_0 = \hat{\Pi}^{\mathsf{b}}_{1,1} = \tilde{\Sigma}^{\mathsf{b}}_{1,1} = \hat{\Sigma}^{\mathsf{b}}_1 = \mathsf{E}_1$. Further these classes would all have the same languages as LinH due to Bennett **??**. By Corollary 2, any formula that TSC_1^1 proves is $\tilde{\Delta}_1^{b,\{2^{p(|id|)}\}}$, that is equivalent to both a $\tilde{\Sigma}^{\mathsf{b}}_{1,1}$ and $\tilde{\Pi}^{\mathsf{b}}_{1,1}$, is computable in SC. So TSC_1^1 By the $\mathsf{SC} \subset \mathsf{LinH}\Delta_0$ is computable in SC. I.e., $\mathsf{SC} = \mathsf{LinH} = \hat{\Sigma}^{\mathsf{b}}_{1,1}$. Call this (*). If we add to TSC_1^1 the defining axioms of #, then TSC_2^1 still prove $\mathsf{LinH} = \hat{\Sigma}^{\mathsf{b}}_{1,1}$. Thus, the $\hat{\Sigma}^{\mathsf{b}}_{1,1}$-formula $U_1(e,x,z)$ would be provably equivalent to a $\hat{\Pi}^{\mathsf{b}}_{1,1}$ formula in TSC_2^1 and then using the result of Lemma 8, TSC_2^1 would show $\mathsf{NP} = \hat{\Sigma}^{\mathsf{b}}_{1,2} = \hat{\Pi}^{\mathsf{b}}_{1,2} = \mathsf{co\text{-}NP}$. We would also have $\hat{\Pi}^{\mathsf{b}}_{1,2} = \tilde{\Pi}^{\mathsf{b}}_{1,2}$. By Corollary 2, the $\hat{\Delta}_1^b$-consequences of TSC_2^1 are SC implying $\mathsf{SC} = \hat{\Pi}^{\mathsf{b}}_{1,2}$. This together with (*) contradicts Lemma 9. □

Theorem 4 *If $\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,1}} = \mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,2}}$ then $I\Delta_0$ does not prove the MRDP Theorem.*

Proof. If $I\Delta_0$ proves the MRDP Theorem, then since S_1 is conservative over $I\Delta_0$ and $S_1 = \cup_i \breve{S}_1^i$, for some $i > 0$, $\breve{S}_1^i$ proves the MRDP Theorem. So by Lemma 10 and Corollary 2, $E_1 = U_1 = \mathsf{LinH} = \hat{\Sigma}^{\mathsf{b}}_{1,1} = \mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,1}}$. As in the proof of the preceding Theorem, S_2^i could then show $\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,2}} = \mathsf{NP} = \hat{\Sigma}^{\mathsf{b}}_{1,2} = \hat{\Pi}^{\mathsf{b}}_{1,2} = \mathsf{co\text{-}NP}$. As we are assuming $\mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,1}} = \mathsf{L}^{\tilde{\Sigma}^{\mathsf{b}}_{i,2}}$, this would mean $\hat{\Sigma}^{\mathsf{b}}_{1,1} = \hat{\Pi}^{\mathsf{b}}_{1,2}$ giving a contradiction of Lemma 9. □

Theorem 5 *For $j > 0$, TSC_1^1 cannot prove $\mathsf{E}(\mathsf{D}\overline{\mathsf{d}})_{j,1} = \mathsf{U}(\mathsf{D}\overline{\mathsf{d}})_{j,1}$.*

Proof. Suppose TSC_1^1 proves $\mathsf{E}(\mathsf{D}\overline{\mathsf{d}})_{j,1} = \mathsf{U}(\mathsf{D}\overline{\mathsf{d}})_{j,1}$. Hence, it proves $\mathsf{E}(\mathsf{D}\overline{\mathsf{d}})_{j,1} = \hat{\Sigma}^{\mathsf{b}}_{j,1} = \mathsf{LinH} = \hat{\Pi}^{\mathsf{b}}_{j,1}$. If we add to TSC_1^1 the defining axioms of #, the resulting theory TSC_2^1 still proves $\mathsf{E}(\mathsf{D}\overline{\mathsf{d}})_{j,1} = \mathsf{U}(\mathsf{D}\overline{\mathsf{d}})_{j,1}$. Then $\hat{\Sigma}^{\mathsf{b}}_{1,1}$-formula $U_1(e,x,z)$ would be provably equivalent to a $\mathsf{U}(\mathsf{D}\overline{\mathsf{d}})_{j,1}$ formula in TSC_2^1. Using replacement in TSC_2^1 $\mathsf{U}(\mathsf{D}\overline{\mathsf{d}})_{j,2} = \hat{\Pi}^{\mathsf{b}}_{1,2}$. Thus, using the result of Lemma 8, TSC_2^1 would show $\mathsf{NP} = \hat{\Sigma}^{\mathsf{b}}_{1,2} \subseteq \hat{\Pi}^{\mathsf{b}}_{1,2} = \mathsf{co\text{-}NP}$. By Corollary 2, the $\hat{\Delta}_1^b$-consequences of TSC_1^1 and TSC_2^1 are SC implying $\mathsf{SC} = \hat{\Sigma}^{\mathsf{b}}_{j,1} = \hat{\Pi}^{\mathsf{b}}_{j,2}$, contradicting Lemma 9. □

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References

[1] J. Bennett. *On Spectra.* Doctoral Dissertation, Princeton University, 1963.

[2] S.R. Buss. *Bounded Arithmetic.* Bibliopolis, Napoli, 1986.

[3] S.R. Buss, J. Krajíček, and G. Takeuti. *Provably total functions in bounded arithmetic theories R_3^i, U_2^i, and V_2^i.* In P. Clote and J. Krajíček, editors, Arithmetic, Proof Theory and Computational Complexity, pages 116–161. Oxford Science Publications, 1993.

[4] Y. Chen, M Müller, and K. Yokoyama. *A Parameterized Halting Problem,* Δ_0 *Truth and the MRDP Theorem.* The Journal of Symbolic Logic, Volume 90, Issue 2, June 2025, pages. 483 – 508 .

[5] S. Cook and P. Nguyen. Logical Foundations of Proof Complexity, Perspectives in Logic (Cambridge University Press. Cambridge, 2010).

[6] P. Clote and G. Takeuti. First order bounded arithmetic and small boolean circuit complexity classes. In P. Clote and J. Remmel, editors, *Feasible Mathematics II*, pages 154–218. Birkhäuser, Boston, 1995.

[7] H. Gaifman and C. Dimitracopoulos. Fragments of Peano's arithmetic and the MRDP theorem. Monographie 30 de L'Enseignement Mathématique, pages 187–206, 1982.

[8] J. Krajíček. *Bounded Arithmetic, Propositional Logic and Complexity Theory.* Cambridge University Press, 1995.

[9] J. Krajíček. Fragments of bounded arithmetic and bounded query classes. *Transactions of the American Mathematical Society*, 338(2):587–598, August 1993.

[10] J. Johannsen and C. Pollett. On the Δ_1^b-Comprehension Rule. In S. Buss, P. Hájek and P. Pudlák *Lecture Notes in Logic 13 – Logic Colloquium 1998*, pages 269–286, A.K. Peters, 2000.

[11] Y. Matiyasevich. Enumerable sets are Diophantine. *Dokl. Acad. Nauk*, 191:279–282, 1970.

[12] V.A. Nepomnjaščiĭ. Rudimentary predicates and Turing computations. *Dokl. Acad. Nauk*, Vol. 195, pages 282–284, 1970, transl. Vol. 11 1462–1465, 1970.

[13] C. Pollett. Structure and definability in general bounded arithmetic theories. *Annals of Pure and Applied Logic*. Vol. 100. pages 189–245, October 1999.

[14] C. Pollett. A Theory for Logspace and NLIN versus co-NLIN. *Journal of Symbolic Logic.* 68(4):1082–1090. December 2003.

[15] G. Takeuti. *RSUV* isomorphisms. In P. Clote and J. Krajíček, editors, *Arithmetic, Proof Theory and Computational Complexity*, volume 23 of *Oxford Logic Guides*, pages 364–386. Clarendon Press, Oxford, 1993.