Using Translations to Separate Bounded Arithmetic Theories

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Goal

Would like to have a Toolkit of methods to separate Bounded Arithmetic theories.

It is interesting to try to separate Bounded Arithmetic Theories since they are often closely related to computational complexity classes. 1 So separating theories might shed some insight into separating classes 2 It is probably easier to separate theories than classes.

In the unrelativized case almost no techniques are known to separate theories.

Upward Translations

$L_2 = \{\leq, 0, S, +, \cdot, 2^{|x||y|}, |x| = \lceil \log_2(x+1) \rceil, \dot{-}, \lfloor \frac{x}{2} \rfloor, \lfloor \frac{x}{2^i} \rfloor\}$

BASIC – open axioms for these symbols

S_2 – BASIC + INDUCTION for bdd formulas.

S_2^0 - BASIC + Length induction for formulas where all quantifiers either $(\forall x \leq |t|)$ or $(\exists x \leq |t|)$

LIND $A(0) \wedge \forall x (A(x) \supset A(sx)) \supset \forall x A(|x|)$

(Takeuti (for a weaker lang), Johannsen)

Can map $S_2^0 \to S_2$

$x \mapsto \langle$ number of 1's in lead block of 'on' bits, number of 0's in next block of 'off' bits $\cdots \rangle$

i.e., $11001 \mapsto \langle 2,2,1 \rangle$

S_2 can define formulas for L_2 - fns on such sequences. Extend to a translation of S_2^0 formulas in S_2.

(Up Cont'd)

Get if $S_2^0 \vdash \varphi(\vec{a})$ then
$$S_2 \vdash PSEQ(a_1) \wedge \cdots \wedge PSEQ(a_n) \rightarrow \varphi^c(\vec{a})$$

Can use to show S_2^0 cannot define $\lfloor \frac{x}{3} \rfloor$.

since if it could $S_2 \vdash PSEQ(x) \rightarrow \exists y \varphi^c_{\lfloor \frac{x}{3} \rfloor}(x,y)$

$\hat{\Sigma}_1^b$ ∴ by Parikh's Thm $\exists y \leq t\, \varphi^c_{\lfloor \frac{x}{3} \rfloor}(x,y)$

on inputs of form $\langle a+1 \rangle$ output $\langle \overbrace{1, \ldots, 1}^{a} \rangle$

code for $2^{a+1}-1$ and code

code for $\underbrace{a \text{ alternations}}_{1010\cdots10}$ for this not bdd by any $t \in L_2$

Johannsen has abstracted a model theoretic proof of this result.

Can also extend this "number of alternations" technique to show the theory

$$Z = \text{BASIC} + \text{pairing} + \hat{\Sigma}_i^b\text{-IND}^{\{\|id\|_{i+3}\}}$$

(Pollett '98) cannot define $\lfloor \frac{x}{3} \rfloor$ with a $\exists x \leq t \forall z \leq |s|$ open formula

S_2 can $\hat{\Sigma}_1^b$ define $\lfloor \frac{x}{3} \rfloor$; however if $Z \vdash PH\downarrow$ $Z = S_2$

so $Z \nvdash PH\downarrow$

Facts about $I\Delta_0 + exp$, $IOpen(exp)$, & $IOpen$

Will now consider how downward translations may be useful to separate theories.

For us $I\Delta_0 + exp := S_2 + \exists z \; x^y$

conservative extension of usual def$^{\underline{n}}$

$I\Delta_0(exp) := S_2 + 2$ axioms defining 2^y

conservative extension of $I\Delta_0 exp$

Define $2^{\min(|y|, x)} := \left\lfloor 2^{|y|} / 2^{|y| \dot{-} x} \right\rfloor$

$IOpen := BASIC + \text{open-}IND$

$IOpen(exp) := BASIC + 2$ axioms defining 2^y

$I\Delta_0(exp)$ not interpretable in S_2

Some facts: $I\Delta_0(exp) = IE_1(exp)$ } Kaye

$I\Delta_0(exp) \vdash MRDP$ } Gaifman Dimitracopoulos

Paris Wilkie Not known if $IOpen(exp)$ has nonstandard recursive models

$\rightsquigarrow I\Delta_0(exp) \nvdash Con(Q)$, however $I\Delta_0(exp) \vdash$ FCFCon(IOpen)

Separation

Formulate IND_A as an inference:

$$\frac{A(x), \Gamma \to \Delta, A(Sx)}{A(0), \Gamma \to \Delta, A(t)}$$

Given $t \in L_2 \cup \{exp\}$ define t^M as

0 if $t = 0$

a if $t = a$

$2^{3|h|^2}$ if t is Sh or 2^h or $|h|$ or $h \dot{-} s$ or $\lfloor h/2^s \rfloor$

$h^M \circ s^M$ if $t := h \circ s$, $\circ \in +, \cdot, \#$

Define t^n as term where a or x replaced with $|X|_n$ (n-lengths of x) and where every $exp(s)$ replaced with $2^{\min(|s^M|, s^n)}$

Extend to formulas in natural way.

Thm: Suppose $IOpen(exp) \vdash A$ an open-formula with free-cut free proof P. Let $n := \max(exp\text{-}rank(P), exp\text{-}rank(P)+1)$

Then $IOpen \vdash A^n$.

Note: such a proof only involves only open formulas.
Not too hard to verify IOpen proves translation of 2^y axioms and open(exp)-IND.

<u>Thm</u> $I\Delta_0(exp) \neq IOpen(exp)$

<u>pf</u> $I\Delta_0(exp) \nvdash FCF\,Con(I\Delta_0(exp))$

However $I\Delta_0(exp) \vdash FCF\,Con(IOpen(exp))$

why? Since any FCFree proof of $0=1$ in $IOpen(exp)$ would involve only open formulas, $I\Delta_0(exp)$ could convert code of such a proof into code of an $IOpen$ proof of $(0=1)^n$ for the appropriate n. But $(0=1)^n := 0=1$ & $I\Delta_0(exp) \vdash FCF\,Con(IOpen)$. ///

Conclusion

- Can add $f^{\underline{n}}$ symbols to lang and still get result provided open axioms for new symbols don't involve f and not fast growing.

  Ex/ $\lfloor x/y \rfloor$

- Might be useful to study downward translations to put limits on how far theories can be separated.

  Map like $x \mapsto |x|^k$ for each k could rule out exponential separations of theories or their propositional translations