Circuit Principles, The Weak Pigeonhole Principle, and RSA

Chris Pollett

CS Colloquium, SJSU

Nov. 4, 2004.

(Joint work with Norman Danner, Wesleyan)

Outline

- Motivation
- Weak Pigeonhole Principles
- Connections to RSA
- Connections to Circuit Lower Bounds
- Some new results

Motivation

1. Answer Clay Math Institute question of whether P = NP, earn a \$1,000,000 + academic glory.
2. If attempt to answer (1) fails, then show major cryptographic algorithm is breakable. Be paid mega-bucks to keep it quiet.

Strategy

- Krajicek and Pudlak [KP98] show there is a polynomial time algorithm that makes use of a black box for injective weak pigeonhole "collisions" that can break the RSA cryptographic scheme [RSA77].
- Jerabek [J04] shows that over certain weak systems of arithmetic the existence of strings hard for circuits of size n^k is equivalent to the provability of the surjective weak pigeonhole principle. So if one could prove the weak pigeonhole principle in these systems one might be one step closer to showing $P \neq NP$.
- These results aren't immediately connected because they use different pigeonhole variants, but maybe finding connections would solve one or the other of the motivational problems.

Weak Pigeonhole Principles

Given a relation R(x,y,z) (sometimes R := f(x,z) = y for some f.)

- iWPHP(R):

  $\forall\ x < n^2\ \exists!\ y < n\ R(x,y,z) \supset$

  $\exists\ x_1,x_2 < n^2\ \exists\ y < n\ [x_1 \neq x_2 \wedge R(x_1,y,z) \wedge R(x_2,y,z)]$

  If R is a function from n^2 into n, it is not one-to-one (two points map to the same value).

- sWPHP(R):

  $\forall\ x < n\ \exists!\ y < n^2\ R(x,y,z) \supset \exists\ y < n^2\ \forall\ x < n \neg R(x,y,z)$

  If R is a function from n into n^2, then it is not onto (some value for y is missed).

- mWPHP(R):

  $\forall\ x < n^2\ \exists\ y < n\ R(x,y,z) \supset$

  $\exists\ x_1,x_2 < n^2\ \exists\ y < n\ [x_1 \neq x_2 \wedge R(x_1,y,z) \wedge R(x_2,y,z)]$

  If R is a multifunction from n^2 into n it is not one-to-one (two points map to the same value).

Relationships between Principles

- Using essentially just logic can show:

  mWPHP(R) $\supset$ iWPHP(R)

  and

  mWPHP(R) $\supset$ sWPHP(R)

- Depending on what formal system you are using it is not known the exact relationship between iWPHP(R) and sWPHP(R). (More on this later)

RSA

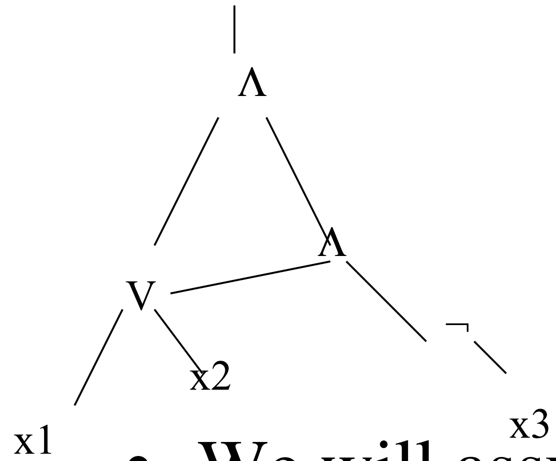
- Public key crypto scheme proposed by Rivest, Shamir, Adleman 1977.
- For this talk, an RSA instance consists of (1) $n=pq$ (where p and q are primes), (2) d and e which are inverse modulo $(p-1)(q-1)$, (3) a message $m < n$ and a ciphertext $c < n$ such that $c \equiv m^e \bmod n$ and $m \equiv c^d \bmod n$.
- Can solve this instance if given n, e, and c one can compute m.

RSA and the iWPHP(f) (Krajicek and Pudlak)

- Assume gcd(c,n) = 1; otherwise, trivial.
- Suppose had a black box that given the function $f(x) = c^x \bmod n$ computes $x_1 < x_2 < n^2$ such that $c^{x1} \equiv c^{x2} \bmod n$. Let $r_0 = x_1 - x_2$.
- Now calculate $r_1 = r_0/\gcd(e, r_0)$ … $r_v = r_{v-1}/\gcd(e, r_{v-1})$ until $r_v = r_{v-1}$ (at most $\log r_0$ steps). Call this last value r. (gcd is p-time using Euclid's Algorithm.)
- If s is order of c mod n, then can show gcd(e,s) = 1. So also have that s divides r_i for each i. Hence s divides r.

More RSA and iWPHP

- Since by construction $\gcd(e, r) = 1$ can using Euclid to get a d' such that $d'e = 1 + tr$.
- Now calculate $c^{d'}$ mod n.
- Done.
- This works since s divides r and $c^{d'} \equiv m^{ed'} \equiv m^{1+tr} \equiv m \bmod n$



Circuits

- We will assume our circuits use AND, OR, and NOT for gates. A family of 0-1 valued circuits $\{F_n\}$ has size less than t(n) if each F_n can be written down as a string of length less than t(n). If t is a polynomial, we say the family $\{F_n\}$ is in P/poly.
- How hard is it to show there is a circuit that requires size n^2?
- Not known if any relation in NP requires n^2 size circuit families. R(x) is an NP relation if R is of the form $\exists$ y, len(y) <p(len(x)) Q(x,y) where Q is p-time computable.

What is a hard relation for circuits of size n^k?

- Consider the p-time function f whose input is a 0-1 valued circuit $C(x_1 \ldots x_n)$ of size $< n^k$ and whose output is a string $S=s_1 \ldots s_m$ where s_i is the output of C on input i (where i is suitably padded with 0's).
- By our definition of size C can be written as a binary string of length $<n^k$. This in turn is a number less than $<2^{n\wedge k}$. If $m=2n^k$, then S is a number $< 2^{2n\wedge k}$, and we can apply sWPHP(f), to get a string which disagrees on some input $i<m$ with any circuit of size n^k.
- Once we know such an S exists we can search for the least such S and use it to get a hard relation.
- Can use this idea to show there are hard relations for n^k sized circuits in NP^{NP}. (There is a slightly stronger result original noticed by Kannan.)

Proving Lower Bounds

- We are interested in how strong a formal system is needed to prove the previous result.
- NP $\not\subset$ P/poly => P $\neq$ NP. If a formal system can't prove lower bounds, it can't prove NP $\not\subset$ P/poly; therefore, P=NP is consistent with the system.
- Understanding why such a consistency might be possible might shed light on how to prove P $\neq$ NP.

Weak Arithmetics

- Have BASIC axioms like:

  $y <= x \supset y <= S(x)$

  $x+Sy = S(x+y)$

  for the symbols 0, S, +, *,$2^{|x||y|}$, $|x|$, -, [x/2^i], <=
- Have IND_m induction axioms of the form:

  $A(0) \wedge \forall x<|t|_m[A(x) \supset A(S(x))] \supset A(|t|_m)$

  Here t is a term made of compositions of variables and our function symbols and $|x|_0=x$, $|x|_m=|\,|x|_{m-1}|$.
- For example, S^1_2 has BASIC axioms together with induction IND_1 axioms for formulas of the form:

  $\exists y<=s\forall z<=|u|A(x,y,z)$ where s,u terms and A is a quantifier free formula. These kind of predicates are exactly the NP ones.
- R^2_2 has BASIC axioms together with induction IND_2 for formulas of the form $\exists y<=s\forall z<=u\ \exists w <=|v|\ A(x,y,z,w)$

Equivalences

- Let $HARD_k$ be the formalization of the statement: "There is a string S of size $2n^k$ which is not computed correctly on all values $<2n^k$ by a circuit of size n^k."
- Let FP be the class of p-time functions. It is open whether S^1_2 can prove sWPHP(FP).
- Jerabek [J04] shows over S^1_2 the statements $HARD_k$ for $k>0$ are equivalent to sWPHP(FP).
- We've essentially seen one direction of this. The idea of the other direction is that given a p-time function for which the sWPHP fails we can find a $n^{k'}$ size circuit computing this function. For any $k>k'$, by iterating this function $O(|n|)$ times, we can get a circuit C' of size $n^{k'+1}$ whose domain is n-bit numbers but whose range is all strings of size $2n^k$. Let C be the circuit which on input $i <2n^k$ and s and an n bit number computes the ith bit of C'. For any fixed S of length $<2n^k$ we can now hard code the s that maps to it in C to get a circuit showing S does not satisfy $HARD_k$.

Towards our results

- As mentioned before the relationship between sWPHP and iWPHP is not known for weak theories like S^1_2.
- The witnessing theorem for S^1_2 says if S^1_2 proves a formula like $\exists y <= s \forall z <= |u| A(x,y,z)$ then there is a p-time function f(x) such that $\forall z <= |u| A(x,f(x),z)$. For R^2_2 the analogous result gives an f contained in quasi-polynomial time.
- Using this Krajicek and Pudlak showed if S^1_2 proves iWPHP(FP) then RSA is insecure against p-time attacks.
- We asked two questions: (1) Can similar results be obtained for sWPHP variants. (2) What happens when take relations in the pigeonhole principles rather than functions.

Our Results I

- Let HardBlks(k) be the formula which says there is a string S of length $2n^k$ such that there is no circuit C(i,s) of size n^k which outputs true iff s is the ith block of n bits from S.
- We show for each k>0, S^1_2 + sWPHP(P^{NP}(log)) proves HardBlks(k).
- On the other hand, $S^1_2 + U_k$ HardBlks(k) proves sWPHP(NP).
- This does not yet give a connection with RSA. For that we needed to look at mWPHP since it implies both iWPHP and sWPHP.

Our Results II

- Given a relation R suppose we know there is a value for y of length < p(x) for some polynomial p such that R(x, y). Could then imagine the relation which computes $R(B(z),y1) \Lambda R(y1, y2) \Lambda \ldots \Lambda R(ym,E(z))$.
- The class Iter(PV,polylog) consists of such relations where R is p-time and iterate at most polylog times.
- Similarly, we define an IterHardBlks(k) which says an iterated circuit of size n^k cannot block recognize some string of size $2n^k$.
- We show R^2_2 proves IterHardBlks(k) is equivalent to mWPHP(Iter(PV,polylog)) and implies iWPHP(FP).
- Therefore, if R^2_2 prove lower bounds for iterated circuits, then RSA is vulnerable to quasi-polynomial time attacks.