Circuit Principles and Weak Pigeonhole Variants

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Outline

- Motivation
- Weak Pigeonhole Principles
- Connections to Circuit Lower Bounds
- Some new results

Motivations for our Paper

- We wanted to understand how much mathematics is needed to show that there exists a set which requires a large circuit.
- Recently, Jerabek [J04] has shown this problem to be connected to the weak pigeonhole principle.
- So we wanted to explore this connection further…

Weak Pigeonhole Principles

Given a relation R(x,y,z) (sometimes R := f(x,z) = y for some f.)

- iWPHP(R):

  $\forall\ x < n^2\ \exists!\ y < n\ R(x,y,z) \supset$

  $\exists\ x_1,x_2 < n^2\ \exists\ y < n\ [x_1 \neq x_2 \wedge R(x_1,y,z) \wedge R(x_2,y,z)]$

  If R is a function from n^2 into n, it is not one-to-one (two points map to the same value).

- sWPHP(R):

  $\forall\ x < n\ \exists!\ y < n^2\ R(x,y,z) \supset \exists\ y < n^2\ \forall\ x < n \neg R(x,y,z)$

  If R is a function from n into n^2, then it is not onto (some value for y is missed).

- mWPHP(R):

  $\forall\ x < n^2\ \exists\ y < n\ R(x,y,z) \supset$

  $\exists\ x_1,x_2 < n^2\ \exists\ y < n\ [x_1 \neq x_2 \wedge R(x_1,y,z) \wedge R(x_2,y,z)]$

  If R is a multifunction from n^2 into n it is not one-to-one (two points map to the same value).

Relationships between Principles

- Using essentially just logic can show:

  $\mathrm{mWPHP}(R) \supset \mathrm{iWPHP}(R)$

  and

  $\mathrm{mWPHP}(R) \supset \mathrm{sWPHP}(R)$

- Depending on what formal system you are using it is not known the exact relationship between iWPHP(R) and sWPHP(R). (More on this later)

Circuits

We say a predicate Y (or function f) is computed by a family of AND, OR, NOT circuits $\{F_n\}$, if F_n correctly computes Y(x) (resp. f(x)) for n bit numbers. If each F_n can be coded as a string of size less than g(n), then we say the circuit family has size g(n).

How hard is it to show there is a predicate that requires n^2-size circuits?

- Not known if any sets in NP requires n^2 size circuit families.
- If we allow sets in harder complexity classes can use Kannan style or fancier arguments.

What is a hard relation for circuits of size n^k?

Consider the p-time function f whose input is an encoding of a 0-1 valued circuit $C(x_1\ldots x_n)$ of size $< n^k$ and whose output is a string $S=s_0\ldots s_{m-1}$ where s_i is the output of C on input i (where i is suitably padded with 0's).

$$\begin{tikzcd}
{} & C(00\ldots0) & C(00\ldots1) \ldots & C(00\ldots(m-1)_2) \\
f: \lceil C \rceil \arrow[r] & s_0 \arrow[u, equal] & s_1 \ldots \arrow[u, equal] & s_{m-1} \arrow[u, equal]
\end{tikzcd}$$

By our definition of size C can be written as a binary string of length $<n^k$. This in turn is a number less than $<2^{n\wedge k}$. If $m=2n^k$, then S is a number $< 2^{2n\wedge k}$, and we can apply sWPHP(f), to get a string which disagrees on some input $i<m$ with any circuit of size n^k.

Hard relation cont'd…

- Once we know such an S exists we can search for the least such S and use it to get a hard relation.
- Can use this idea to show there are hard relations for n^k sized circuits in NP^{NP}. (There is a slightly stronger result original noticed by Kannan.)
- Well studied weak theories of arithmetic such as S^3_2 or T^2_2 can carry out this argument.

Remembering what we are doing

- We are interested in how strong a formal system is needed to prove the previous result. (Lower bound).
- NP $\not\subset$ P/poly => P $\neq$ NP. If a formal system can't prove lower bounds, it can't prove NP $\not\subset$ P/poly; therefore, P=NP is consistent with the system.
- Understanding why such a consistency might be possible might shed light on how to prove P $\neq$ NP.

Formal Systems

- Have BASIC axioms like:

  $y \leq x \supset y \leq S(x)$

  $x+Sy = S(x+y)$

  for the symbols 0, S, $+$, $\cdot$, $x\#y := 2^{|x||y|}$, $|x| :=$ length of x, $\dot{-}$, $\lceil x/2^i \rceil$, $\leq$

- Have IND_m induction axioms of the form:

  $A(0) \wedge \forall\, x<|t|_m[A(x) \supset A(S(x))] \supset A(|t|_m)$

  Here t is a term made of compositions of variables and our function symbols and $|x|_0=x$, $|x|_m=|\,|x|_{m-1}|$.

Formal Systems cont'd

- A Σ^b_i-formula is a formula of the form:

  $$\exists x_1 \le \underbrace{t_1 \forall x_2 \le t_2 \cdots Qx_i \le t_i\, Qx_{i+1}}_{\text{i+1 alternations, innermost begin length bounded}} \le |t_{i+1}| A$$

  where A is an open formula. A Π^b_i-formula is defined similarly but with the outer quantifier being universal.

  T^i_2 is the theory BASIC + Σ^b_i-IND_0

  S^i_2 is the theory BASIC + Σ^b_i-IND_1

  R^i_2 is the theory BASIC + Σ^b_i-IND_2

- These theories are well-studied and are known to be closely connected with computational complexity.
- If we add to the language a function symbol $x\#_3y$ with $|x\#_3y|=|x|\#|y|$, then get theories T^i_3, S^i_3, R^i_3.

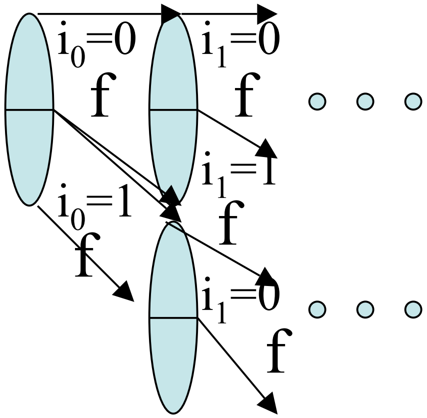
Formalizing Hard Sets

- Let $HARD_k$ be the formalization of the statement: “There is a string S of size $2n^k$ which is not computed correctly on all values $<2n^k$ by a circuit of size n^k.”
- Let PV be the class of p-time functions (actually PV is really the function symbols plus defining axioms for these functions). It is open whether S^1_2 can prove sWPHP(PV).
- Jerabek [J04] shows over S^1_2 the statements $HARD_k$ for $k>0$ are equivalent to sWPHP(PV).

Intuition behind Jerabek

- We’ve already seen sWPHP(PV) $\supset$ $HARD_k$.
- For $HARD_k$ $\supset$ sWPHP(PV), suppose there is a p-time function f for which the sWPHP fails.
- Then there is a $n^{k'}$ size circuit family $\{C^f_n\}$ computing this function for some k'. Can iterate f according to a string $i_0 i_1 \cdot\cdot$

Input: 2n bit string. $(2^{|x|})^2 = 2^{2|x|}$, n=|x|



For any k>k' , iterating C^f_n O(|n|) times, we can get a circuit C' of size $n^{k'+1}$ whose domain is $|2n^{k-1}|$ x 2n-bit numbers but whose range is all strings of size $2n^k$.

Let C be the circuit which on input i $<2n^k$ and s, a 2n bit number, computes the ith bit of C'. For any fixed S of length $<2n^k$ we can now hard code the s that maps to it in C to get a circuit showing S is not the hard string of $HARD_k$.

Towards our results

- As mentioned before the relationship between sWPHP and iWPHP is not known for weak theories like S^1_2.
- The witnessing theorem for S^1_2 says if S^1_2 proves a formula like $\exists y \leq s \forall z \leq |u| A(x,y,z)$ then there is a p-time function f(x) such that $\forall z \leq |u| A(x,f(x),z)$. For R^2_3 the analogous result gives an f contained in quasi-polynomial time.
- Using the witnessing theorem, Krajicek and Pudlak showed if S^1_2 proves iWPHP(FP) then RSA is insecure against p-time attacks.
- We asked two questions:
  1. Can one get a connection between circuit lower bound provability and RSA? Idea: Try mWPHP use pre-images.
  2. What happens when one takes relations in the pigeonhole principles rather than functions? Using pre-images will force us to answer this anyway in the mWPHP setting.

Our Results I

- Let HardBlks(k) be the formula which says there is a string S of length $2n^k$ such that there is no circuit C(i,s) of size less than n^k which outputs true iff s is the ith block of n bits from S.
- We use blocks now because we want to look at the relational case. Given a relation R(x,y), to find a y from x such that R(x,y) might not be easy for our theories.
- We show for each k>0, S^1_2 + sWPHP(P^{NP}(log)) proves HardBlks(k). Here P^{NP}(log) is the class of p-time relations allowed O(log n) queries to an NP oracle .
- On the other hand, $S^1_2 + U_k$ HardBlks(k) proves sWPHP(NP).
- This does not yet give a connection with RSA. For that we needed to look at mWPHP since it implies both iWPHP and sWPHP.

Our Results II

- Given a relation R suppose we know there is a value for y of length < p(|x|) for some polynomial p such that R(x, y). Could then imagine the relation which computes $R(B(z), y_1) \Lambda R(y_1, y_2) \Lambda\ldots \Lambda R(y_m, E(z))$.
- The class Iter(PV,polylog) consists of such relations where R is p-time and iterate at most polylog times. This is a slightly stronger class than the p-time relations. It is weaker than the NP relations and is a little similar to local search classes previously considered by others.
- Similarly, we define an IterHardBlks(k) which says an iterated circuit of size n^k cannot block recognize some string of size $2n^k$.
- We show R^2_2 proves IterHardBlks(k) is equivalent to mWPHP(Iter(PV,polylog)) and implies iWPHP(PV).
- Therefore, if R^2_3 proves lower bounds for iterated circuits, then RSA is vulnerable to quasi-polynomial time attacks. Hence, if any weaker theor such as $R^2_{2,}$ S^1_2 proves it, RSA would also be insecure.

Conclusion

- Since RSA is considered hard above seems to suggest R^2_3 cannot prove IterHardBlks(k) and this in turn seems to imply it can't prove qNP $\not\subset$ qP/poly.
- On the other hand, it is known by Paris, Wilkie, and Woods that T^2_2 can prove mWPHP(Σ^b_1). So RSA is insecure against the NP-definable multifunctions of this theory (a class called GLS[†]).
- It would be cool to understand what happens for $S^2_{2.}$ Its NP-definable multifunctions are projections of PLS (polynomial local search) problems.

Appendix RSA

- Public key crypto scheme proposed by Rivest, Shamir, Adleman 1977.
- For this talk, an RSA instance consists of (1) $n=pq$ (where p and q are primes), (2) d and e which are inverses modulo $(p-1)(q-1)$, (3) a message $m < n$ and a ciphertext $c < n$ such that $c\equiv m^e \bmod n$ and $m \equiv c^d \bmod n$.
- Can solve this instance if given n, e, and c one can compute m.

RSA and the iWPHP(f) (Krajicek and Pudlak)

- Assume gcd(c,n) = 1; otherwise, trivial.
- Suppose had a black box that given the function $f(x) = c^x \bmod n$ computes $x_1 < x_2 < n^2$ such that $c^{x1} \equiv c^{x2} \bmod n$. Let $r_0 = x_1 - x_{2.}$
- Now calculate $r_1 = r_0/\gcd(e, r_0)$ … $r_v = r_{v-1}/\gcd(e, r_{v-1})$ until $r_v = r_{v-1}$ (at most log r_0 steps). Call this last value r. (gcd is p-time using Euclid’s Algorithm.)
- If s is order of c mod n, then can show gcd(e,s) = 1. So also have that s divides r_i for each i. Hence s divides r.

More RSA and iWPHP

- Since by construction gcd(e, r) = 1 can using Euclid to get a d' such that d'e = 1 + tr.
- Now calculate $c^{d'}$ mod n.
- Done.
- This works since s divides r and $c^{d'} \equiv m^{ed'} \equiv m^{1+tr} \equiv m$ mod n