Bounded Query Classes

&

Bounded Arithmetic

Overview

1. Old motivating results. (Bounded Arithmetic)
   (a) Definability & Structural Results
   (b) Equalities $\Rightarrow PH\downarrow$

2. New Results
   (a) Definability & Structural Results
   (b) Weak Equalities $\Rightarrow PH\downarrow$

3. Models Separating Theories (relativized)

Bounded Arithmetic Theories

We'll work in the language:

$L_2 := \{0, Sx := x+1, +, \cdot, \leq, \dot{-}, |x|, \lfloor \frac{x}{2^i} \rfloor,$
$x \# y = 2^{|x||y|}\}$

BASIC := Set of $^{33}_{\wedge}$ open axioms for these symbols.

Example axioms
$0 \leq x$
$x \leq x+y$
$\vdots$

BA Hierarchy

$\Sigma^b_0 = \Pi^b_0$ - formulas w/ all quantifiers of form $\exists x \leq |t|$ or $\forall x \leq |t|$

$\Sigma^b_i \supset \Pi^b_{i-1}$ - closed under $\wedge, \vee, \forall x \leq |t|, \exists x \leq t$

$\Pi^b_i \supset \Sigma^b_{i-1}$ - closed under $\wedge, \vee, \forall x \leq t, \exists x \leq |t|$

$\Sigma^b_i, \Sigma^p_i, \hat{\Sigma}^b_i$ - define same rel'ns

$\Pi^b_i, \Pi^p_i, \hat{\Pi}^b_i$ - define same rel'ns

Here a $\wedge$ indicates prenex normal form formulas

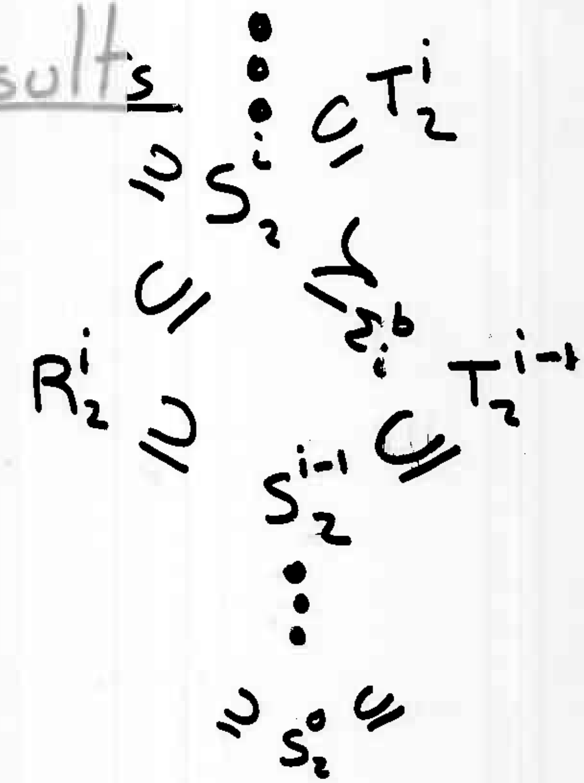
Ψ-IND$^{\tau}$: $\alpha(0) \wedge \forall x(\alpha(x) \supset \alpha(Sx)) \supset \forall x \alpha(l(x))$
where $\alpha \in \Psi$, $l \in \tau$

$T^i_2 := \text{BASIC} + \Sigma^b_i\text{-IND}^{\{id\}}$ where $id(x) = x$

$S^i_2 := \text{BASIC} + \Sigma^b_i\text{-IND}^{\{|id|\}}$

$R^i_2 := \text{BASIC} + \Sigma^b_i\text{-IND}^{\{||id||\}}$

Old Results



(a) $S_2^{i+1} = T_2^i \Rightarrow \Sigma_{i+2}^p = \Pi_{i+2}^p$ i.e., PH$\downarrow$ (KPT)

(b) $S_2^{i+1}(\alpha) \supsetneq T_2^i(\alpha) \supsetneq S_2^i(\alpha)$ (KPT, Krajicek)

Def$^{\underline{n}}$ (a) A multifunction is a total rel$^{\underline{n}}$

(b) T can Ψ-define multif$^{\underline{n}}$ $f(x)=y$ if $T \vdash \forall x \exists y A_f(x,y)$ where $A_f \in \Psi$. & $\mathbb{N} \models A_f(x,y) \Leftrightarrow f(x)=y$.

(c) For f$^{\underline{ns}}$ T must prove y unique.

(d) A is Δ_i^b in T iff $T \vdash A \Leftrightarrow A^\Sigma \Leftrightarrow A^\Pi$ where $A^\Sigma \in \Sigma_i^b$, $A^\Pi \in \Pi_i^b$.

<u>FACTS:</u> (1) Σ_{i+1}^b-definable f$^{\underline{ns}}$ of S_2^{i+1} & T_2^i are $FP^{\Sigma_i^p}$ (Buss)

(2) Σ_{i+1}^b-definable f$^{\underline{ns}}$ of S_2^i are $FP^{\Sigma_i^p}$(wit, log) (Kraj)

(3) Σ_1^b-definable f$^{\underline{ns}}$ of R_2^1 are FNC. (Allen, Clote-Takeuti)

(4) For Δ_i^bness drop F and wit to get result.

Questions

- Does the trend $FP^{\Sigma_i^p}$ for T_2^i, $FP^{\Sigma_i^p}(\text{wit}, \log)$ for S_2^i continue to $FP^{\Sigma_i^p}(\text{wit}, \log\log)$ for R_2^i?

- Is $S_2^i \preceq_{\Sigma_{i+1}^b} R_2^{i+1}$? What makes one bounded arithmetic theory conservative over another? Can the $T_2^i \preceq_{\Sigma_{i+1}^b} S_2^{i+1}$ result be improved?

- Can the $T_2^i = S_2^{i+1} \Rightarrow \Sigma_{i+2}^p = \Pi_{i+2}^p$ result be improved? Does $S_2^i \subsetneq S_2^{i+1}$ $\forall i$ imply anything about PH?

- What relativized separations occur between these theories? From (Krajicek, KPT) know

$$S_2^i(\alpha) \subsetneq T_2^i(\alpha) \subsetneq S_2^{i+1}(\alpha)$$

- $\exists M$ $\quad M \models S_2^i(\alpha)$ $\forall i$ | By their method of pf

Since Δ_{i+1}^b-predicates not same $\quad M \models S_2^i(\alpha) \neq S_2^{i+1}(\alpha)$

$\quad M \models PH\uparrow$

Does $\exists M$ $\quad M \models S_2^i(\alpha)$ $\forall i$

$\quad M \models S_2^i(\alpha) \neq S_2^{i+1}(\alpha)$ $\forall i$

$\quad M \models PH\downarrow$?

Prenex Theories

EBASIC := BASIC + 3 open axioms for pairing

Remark: EBASIC $\subseteq R^0_2$

Consider theories with axiom schema using prenex formulas.

advantage formulas appearing is not free pf simpler

$$\hat{T}^{i,\tau}_2 := \text{EBASIC} + \hat{\Sigma}^b_i\text{-IND}^{\tau}$$

$$C^{i,|\tau|}_2 := \text{EBASIC} + \text{open-IND}^{|\tau|} + \hat{\Pi}^b_i\text{-REPL}^{|\tau|}$$

where $|\tau| = \{|\ell| : \ell \in \tau\}$ and $\hat{\Pi}^b_i\text{-REPL}^{|\tau|}$ is

$$\forall x \leq |\ell(s)|\, \exists y \leq t\, \alpha(x,y) \Leftrightarrow \exists w < 2^{(t^* \# \ell(s))}\, \forall x < |\ell(s)|\, \alpha(x, \dot{\beta}(x, |t^*|, w))$$

<u>Thm</u> (1) $\hat{T}^{i,\{id\}}_2 = T^i_2$, $\hat{T}^{i,\{|id|\}}_2 = S^i_2$

(2) $R^i_2 = \hat{T}^{i,\{||id||\}}_2 + \hat{\Pi}^b_{i-1}\text{-REPL}^{\{|id|\}}$

(3) $\hat{T}^{i,\{||id||\}}_2 \preceq_{B(\hat{\Sigma}^b_{i+1})} R^i_2$

(4) $\hat{T}^{i,|\tau|}_2 \preceq_{B(\hat{\Sigma}^b_{i+1})} C^{i,|\tau|}_2$ Proof theoretic pf

General Definability Result

<u>Thm</u> The $\hat{\Sigma}^b_{i+1}$-definable multifunctions of $\hat{T}^{i,\tau}_2$ are precisely $FP^{\Sigma^p_i}(wit, |\tau|)$.
The $\hat{\Delta}^b_{i+1}$-predicates are $P^{\Sigma^p_i}(|\tau|)$.

<u>Two Key pts of pf</u>: (1) Define a class of machines $F[|\tau|]^{\Sigma^p_i}_2(wit)$ which have run-times bdd by $|l(h)|$ for $l \in \tau$, $h \in L_2$ that is equivalent to $FP^{\Sigma^p_i}(wit, |\tau|)$. Use techniques of BKT to show $T^{i,\tau}_2$ can define this class.
(2) Since $\hat{T}^{i,\tau}_2$ is a prenex theory pfs of $\Gamma \rightarrow \Delta$ when Γ, Δ have $\hat{\Sigma}^b_i$ formulas contain only $\hat{\Sigma}^b_i$-formulas. So witness argument easier.

Other interesting thing... As perform witnessing argument keep witnessing function in a particular syntactic form. As consequences...

(1) $\hat{T}^{i,\tau}_2$ proves its $\hat{\Delta}^b_{i+1}$-predicates equivalent to formulas in $(\exists x < l)(A(x,a) \wedge \neg B(x,a))$ where $l \in \tau$, $A, B \in \hat{\Sigma}^b_i$

(2) Proof theoretic pf that $\hat{T}^{i,|\tau|}_2 \vdash \hat{\Delta}^b_{i+1}\text{-IND}^{|\tau|}$

In particular $S^i_2 \vdash \Delta^b_{i+1}$-LIND

Applications

(1) $\hat{R}^i_2$'s $\hat{\Sigma}^b_{i+1}$-definable f$^{\underline{ns}}$ are $FP^{\Sigma^p_i}$(wit, log lg)

(2) EBASIC's $\hat{\Sigma}^b_{i+1}$-definable multifunctions are $FP^{\Sigma^p_i}(\mathrm{wit}, O(1))$. $\hat{\Delta}^b_{i+1}$ predicates are $P^{\Sigma^p_i}(O(1))$.

Similar to what Herbrand's Thm gives but w/ a witnessing argument.

(3) $\hat{T}^{i,\tau}_2$'s $\hat{\Sigma}^b_{i+k+1}$-definable multif$^{\underline{ns}}$ are precisely $FP^{\Sigma^p_{i+k}}(\mathrm{wit}, O(1))$. $\hat{\Delta}^b_{i+k+1}$-predicates are $P^{\Sigma^p_{i+k}}(O(1))$.

(4) The following equalities imply the collapse of PH to $BH(\Sigma^p_{i+2})$ for purely complexity reasons ([redacted] BCO'93, BF'96, HHH'97)

(1) $T^i_2 = \hat{T}^{i+1, |\tau|}_2$

(2) $T^i_2 = \hat{C}^{i+1, |\tau|}_2$

(3) $\hat{C}^{i, |\tau|}_2 = \hat{T}^{i+1, |\tau|}_2$

| τ containing an unbound term ℓ.

(5) Can construct an oracle (cpollett '97) separating $P^{\Sigma^p_i(A)}(\{\|\ell\|\})$ from $P^{\Sigma^p_i(A)}(\{\|\ell\|^2\})$ and deduce many relativized separations of theories.

Conservation

Would like a general condition for one bdd arithmetic theory to be conservative over another. $T^i_2 \preceq_{\Sigma^b_{i+1}} S^{i+1}_2$ since T^i_2 can
(Buss '91)
simulate S^{i+1}_2 pf of Σ^b_{i+1}-formulas using a witnessing argument. This requires simulating Σ^b_{i+1}-LIND inferences, which requires showing T^i_2 can prove its Σ^b_{i+1}-definable fns are closed under BPR. In $T^{i+1,|\tau|}_2$ case can define a notion of $BPR^{|\tau|}$. Can define a notion of closure under smash for a set of term τ denoted $\tau^{\#}$. Ex $\tau^{\#}$ of $\tau = \{id\}$ is $\{2^{p(||id||)}\}$.

Thm $T^{i,\tau^{\#}}_2 \preceq_{B(\hat{\Sigma}^b_{i+1})} \hat{T}^{i+1,|\tau|}_2$

B comes from fact that for prenex pfs $EBASIC \vdash A(\vec{a}) \Leftrightarrow (\exists w < t_A) Wit^{i+1}_A(w, \vec{a})$ & techniques of Buss '91.

Applications of General Conservation Result

(1) $T_2^i \preceq_{B(\hat{\Sigma}^b_{i+1})} S_2^{i+1}$

(2) $\hat{T}_2^{i,\{2^{p(||x||)}\}} \preceq_{B(\hat{\Sigma}^b_{i+1})} R_2^{i+1}$

(3) $\cdots\ \hat{T}_2^{i-2,\{2^{p_1(2^{p_2(|x|_n)})}\}} \preceq_{B(\Sigma^b_{i-1})} T_2^{i-1,\{2^{p(|x|_n)}\}} \preceq_{B(\Sigma^b_i)} T_2^{i,\{\log}$

while terms still in L_2. and $i>0$

(4) Related to question does $\exists M$ $M \models S_2^i(\alpha)\ \forall i$, $M \models S_2^i(\alpha) \neq S_2^{i+1}(\alpha)$, $M \models PH(\alpha)\downarrow$

<u>Thm</u> $\exists M\ \forall i.$ $T_2^{i,\{|id|_i\}}(\alpha)$ & $T_2^{i,\{|id|_{i-1}\}}(\alpha)$ are modelled by M, $M \models T_2^{i,\{|id|_i\}}(\alpha) \neq T_2^{i,\{|id|_{i-1}\}}$ $M \models PH(\alpha) = \Delta_2^p(\alpha)$.

pf Idea $\Delta_2^b(\alpha)$-predicates of $T_2^{i,\{|id|_i\}}$ contained in $P^{\Sigma_i^{p(\alpha)}}(\log^2)$ whereas those of $T_2^{i,\{|id|_{i-1}\}}$ contain $P^{\Sigma_i^p}(\alpha)$.

It is known (Mocas '93) $P^{\Sigma_i}(n^k) \subsetneq NEXP$ relativizes & by Burhman Torenvliet there is an oracle s.t. $P^{\Sigma_i^p(X)} = NEXP(X)$.

Conjecture

Attack on S_2 problem not too far beyond current technology.

(1) Get an oracle s.t. $PH\downarrow$ and UPLS $\neq$ FP.
(Fenner '98) UniquePLS

(2) Feirrara '95 give a characterizations of Σ_1^b-consequences of T_2^2 using Herbrand techniques. Get an oracle s.t. $PH\downarrow$ and F's class $\neq$ PLS.

(3) Herbrand approach probably generalizes to higher i. Call these classes F_i. Need oracle s.t. $F_i \neq F_{i-1}$ get $PH\downarrow$.