Homework 1

SJSU Students

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1 Problem 5.1-2

Assumption:
We assume that every value returned by RANDOM(0, 1) is independent. Thus we can make repeated calls to get a stream of 1's and 0's. We can thus build a RANDOM(a, b) which makes use of this stream to return a random number between a, and b(inclusive). Since individual bits of the result are random, repeated calls to RANDOM(a, b) will return numbers which will be independent of each other.

$RANDOM(a, b)$

1. $range \leftarrow (b - a)$
2. $nbits \ \leftarrow 1 \ + \ \log_2(range)$
3. $for \ i \ \leftarrow 0 \ to \ nbits$
4. $\quad do \ stream \leftarrow \ append(stream, \ RANDOM(0, 1))$
5. $randNum \ \leftarrow \ parseInt(stream)$
6. $if \ (randNum \ > \ range)$
7. $\quad return \ RANDOM(a, b)$
8. $return \ a \ + \ randNum$

Expected Running Time:
The expected runtime time will be the runtime for the body of the procedure times the expected number of recursions. The number of for loop calls to RANDOM(0,1) is $\lceil \log_2(b-a) \rceil$. (Hopefully, we don't exceed the runtime stack with the recursions; otherwise, we should probably have used a while loop.) The probability that we succeed is given by

$$p = \frac{b-a}{2^{\lceil \log_2(b-a) \rceil}}.$$

The probability we fail is $q = 1 - p$. This can be viewed as a Bernoulli trial, and so the expected number of trials until success goes is $1/p$. Since p is a constant between 1/2 and 1, the expected increase in runtime cause by the recursion is less than a factor of 2. So the expected runtime of the algorithm is $O(\lceil \log_2(b-a) \rceil)$.

2 Problem 5.1-3

Procedure BIASED-RANDOM has outputs with Pr(1) = p and Pr(0) = 1 – p, 0 < p < 1.

Let consider the outcome of 2 consecutive outputs by this procedure: 00, 01, 10, 11 in which they may not happen equally. However, even though the output is biased (i.e. Pr(0) != Pr (1)), the probabilities of 01 or 10 are the same:

Pr(01) = Pr(0)*Pr(1) = p(1-p)
Pr(10) = Pr(1)*Pr(0) = (1-p)p

Hence, by using these 2 outputs with the same probability we can produce unbiased output. Let say when we call BIASED-RANDOM twice, if we get 01 then the answer is 0, and if we get 10 then the answer is 1. We redo the step if getting 00 or 11 since these are biased.

Here is the exact algorithm pseudocode:

UNBIASED-RANDOM
while not found yet

```
output1 = BIASED-RANDOM()
output2 = BIASED-RANDOM()
if output1==0 and output2==1
   then return 0
else if output1==1 and output2==0
   then return 1
```

As we can see here, the expected running time of this algorithm depends on when 01 or 10 comes out. If either 00 or 11 come out, then it must loop back and redo it again until it gets 01 or 10.

Let consider Bernoulli distribution of 2 consecutive outputs of BIASED-RANDOM in which it succeeds when getting either 01 or 10 and fails when getting 00 or 11 with Pr(success) = p(1-p) as above. Then the distribution X denotes the total number of such tries is a geometric distribution.

From the book appendix C.31, we can have:

E[X] = 1 / p(1-p)

This is exactly the expected running time of our UNBIASED-RANDOM function.

3 Problem 5.2-4

Let X be the random variable denoting number of customers that get back their own hat. We want to compute *E[X]*. Also, let $X_1, X_2, ..., X_i, ..., X_n$ be the indicator random variables to indicate that the *ith* customers get their own hats back respectively. More precisely, let X_i be 1, if *ith* customer gets his own hat back, and 0, otherwise.

The number of customers who get their own hat is the sum of these indicators:

$$X = X_1 + X_2 + X_i + \ldots + X_n$$

By taking the expected value of both sides and applying linearity of expectation, we get:

$$E[X] = E[X_1+X_2+\ldots+X_i+\ldots+X_n = E[X_i]+E[X_2]+\ldots+E[X_i]+\ldots+E[X_n].$$

By definition of an indicator random variable and by Lemma 5.1, we have: $E[X_i] = Pr\{ith \text{ candidate got his own hat}\}$, i.e., $Pr(X_i = 1)$ is $\frac{1}{n}$. Therefore,

$$E[X] = \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n}$$

or $E[X] = \sum_{i=1}^{n} n \times \frac{1}{n} = 1$.

So, the expected number of customers who that get back their own hat is 1.

4 Problem 5.4-2

This problem is similar to the birthday problem. However, instead of 365 days, we have b number of bins; and instead of looking for collision amongst days, we look for collisions amongst bins. Therefore, if k is the number of tosses, then from page 108 in the book, the expected number of pairs of balls sharing the same bin grows as

$$\frac{k(k-1)}{2b}$$

So when $k \geq \sqrt{2b} + 1$ this expected number of pairs of balls which end up in the same bin is 1.