More Completeness

CS254

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Outline

- Complete problems for NP

NAE-SAT

- NAE-SAT is the variation on SAT where we ask for a truth assignment which for every clause we have that there are two literal are given different values.

Thm. NAE-SAT is NP-complete

Proof. Look at the reduction of CIRCUIT-SAT to SAT. It actually reduces to NAE-SAT provided we add to one and two variables clauses a new dummy variable z. For instance, for AND of two subcircuits we might have 3 output clauses with a new variable z: $(g_i \vee \neg g_k \vee \neg g_j)$ $(\neg g_i \vee g_k \vee z)$ $(\neg g_i \vee g_j \vee z)$. Suppose there was an assignment A that satisfied the original sets of clauses. We could do this and set z to false. Now the complementary assignment satisfies all the clauses involving inputs since z will be true. As each of the other clauses comes from translating a ↔ to 3CNF (as for example our AND gate above) one can also see the complementary assignment satisfies all clauses.. And if we look at the clause we have at least one literal in each must be false.

Independent Set

- Given a undirected graph G=(V,E), a set I of vertices is called **independent** if for every pair x,y in I there is no edge between them in G.
- The problem INDEPENDENT SET is given a graph G and an integer k, is there a independent ste of size k in G?

Thm. INDEPENDENT SET is NP-complete.

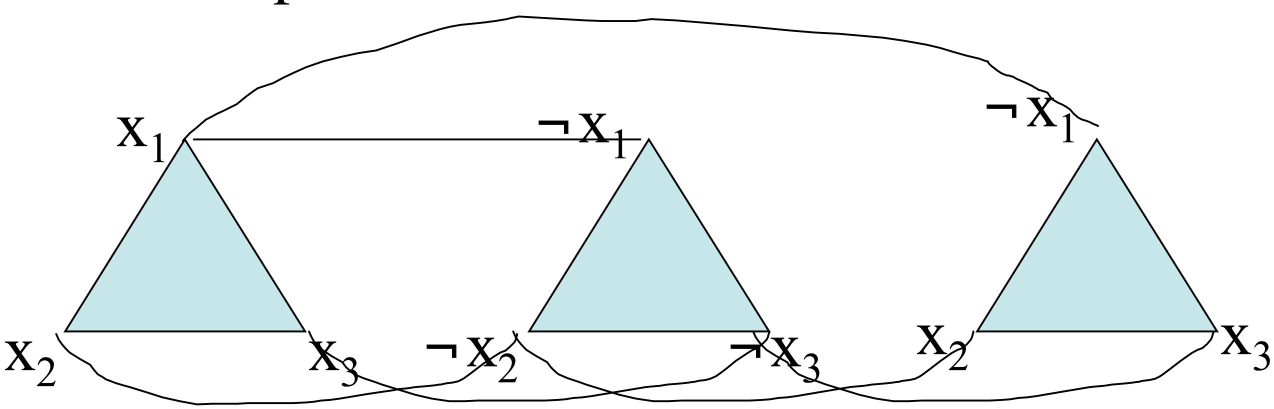
Proof. First the problem is in NP because we can always guess a set of size I and check if it is independent or not. To see its complete we will reduce 3SAT to it. We make use of a graph consisting of a collection of triangle "gadgets", one for each clause. The vertices of a triangle are labelled with the literals of the clauses. We also have between triangles edges between each variable and its negation:

Proof cont'd

A formula like

$(x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_1 \lor x_2 \lor x_3)$

would map to:



Proof Cont'd some more

- Given an instance F of 3SAT we construct such a graph G and write down the integer k = to the number of clauses of the 3SAT instance.
- The claim is that the 3SAT instance F is satisfiable iff G has a independent set of size k.
- Notice in any independent set we can have at most a variable or its negation.
- Since the size of the set is k we must have at least one node from each triangle.
- So imagine the truth assignment where we make the variables for the nodes of the independent set true and all other nodes false.
- This would satisfy each clause of the 3SAT instance.
- On the other hand, suppose F were satisfiable. Then if we pick one true literal from each clause we get an independent set in G.

CLIQUE

- A **clique** in an undirected graph $G=(V, E)$ is a subset $V' \subseteq V$ of vertices each pair of which is connected by an edge.
- *CLIQUE* $=\{\langle G, k\rangle \mid G$ is a graph with a clique of size $k\}$

Theorem. *CLIQUE* is **NP**-complete.

Proof. First *CLIQUE* is in **NP** because we can just guess a set of more than k vertices and check for each possible edge. To see it is NP-complete one can reduce INDEPENDENT SET to it. Namely, given an instance (V,E),k of INDEPENDENT SET we output the CLIQUE instance (V, VxV -E), k.

VERTEX-COVER

- A **vertex cover** (book calls node cover) in an undirected graph G=(V, E) is a subset of the vertices V' such that each vertex in V is connected to a vertex in V' by an edge.
- VERTEX-COVER={<G, k>: graph G has a vertex cover of size k}

Theorem. VERTEX-COVER is **NP**-complete.

Proof. To see it is in **NP** notice if we guess a set of k edges we can check if it is a vertex in polynomial time. To see it is **NP**-complete we reduce *CLIQUE* to this problem. Let $\overline{G}$ denote the complement of a graph G=(V, E), that is, the graph with the same vertices, but with edges {i, j} iff {i, j} is not an edge of G. Then notice G has a clique of size k iff $\overline{G}$ has a vertex cover of size |V|-k.