Reductions

CS254
Chris Pollett
Oct 16, 2006.

Outline

- Complete problems for P and NP

Exhibiting Complete Problems

- We would like to show there are problems complete for P and NP.
- To do this we are going to use the so-called **table method**.
- Let M be a p-time TM which decides L. Let's assume with at most a polynomial slowdown M has just one tape.
- M's computation on input x can be thought of as a $|x|^k$ by $|x|^k$ table where the ith row is the contents of the the first $|x|^k$ tape square at time i. We assume M run in time $|x|^k$. The square j where the tape head at time i is a pair (square_value, state).
- We assume that if M halts at time i then i and all the rows after it are identical.

The Circuit Value Problem

- Recall the circuit value problem (CVP) is given a circuit with all inputs set to true or false, determine if it evaluates to 1 or note.

Theorem. CVP is P-complete with respect to lospace reductions.

Proof. Let M be a decider for a P time language. For some input x, let T_{ij} denote the table (i,j) entry of the last slide. These entries value can be encoded using a fixed number of bits. Given x, our reduction first computes the encodings $T_{1j}..T_{1|x|}{}^k$. Next we output fixed circuits C_{ij} , $2\leq i\leq|x|^k$, $1 \leq j\leq|x|^k$ which can be used to compute the encoding of the of T_{ij} based on the encodings of $T_{i-1j-1}T_{i-1j}T_{i-1j+2}$. Without loss of generality we can assume M always decides its output on the first tape square. So we have the final output of our circuit be the the yes or no value of this tape square in the $|x|^k$ row.

Corollary

- Let MONOTONE CIRCUIT VALUE (MCVP) be the same problem a CVP except where the gates can now only be AND, OR.

Cor. MCVP is P-complete.

Proof. Given an instance of CVP, if the instance was a formula, one could imagine repeatedly apply DeMorgan's laws until all negations have been pushed to the leaves. Since the leaves all have values true or false. We could then easily rewrite expressions such as $\neg$true as false, and $\neg$false as true to get an equivalent monotone circuit. For a circuit some paths from the final gate to a node might have an even number of negations some might have an odd number of negations. So in the circuit case we copy each gate other than the final gate in the input non-monotone circuit to two distinct gates in the output circuit, one to be used as a child for paths through through the circuit with an even number of negations, the other to be used in paths with odd numbers of negations. If the gate is an input in the odd case flip our sign. If the gate is a NOT gate in both cases, we replace the gate with an AND gate with both children being the child of the NOT gate for the opposite cases. If the gate is a AND gate we change it to an OR gate for the odd case. Similarly, if the gate was an OR we change it to an AND for the odd case.

Cook's Theorem

Thm. SAT is NP-complete.

Proof. By introducing extra variables for each gate one can show that CIRCUIT-SAT logspace reduces to SAT. So it suffices to show CIRCUIT-SAT is NP-complete. Let L in NP be decided by $M=(K,\Sigma,\Delta, s)$ in time n^k. We assume M uses only one tape. We assume using the HW that in any given state we exactly two next transitions. So on input x, at step i, $1\leq i\leq|x|^{k,}$ we always have two next choice and we can have a variable c_i to represent which choice we follow. So on input x, we output a circuit as in the CVP case, except now the circuit C_{ij} depends not only on the output of $T_{i-1j-1}T_{i-1j}T_{i-1j+2}$ but also on the value c_{i-1}. The output of this circuit is satisfiable iff there is some setting of the c_i's such that M on that computation path accepts.